

Complete Research Edition

Mathematics of Sound

CliDeck Research Notes · Bass Plus · Version 1.0 · 18 October 2026

This edition collects the illustrated report, its companion articles, the full mathematical appendix, and an English catalogue of ninety exploratory approaches. The downloadable archive also contains the original research files, exact checks, figure data, and website publication instructions.

Guide to this edition

Main report: The Mathematics of Sound and the Search for Negative Peaks, with eight figures.

Companion articles: Sampling and Hidden Peaks; Waves and Lattice Particles; Certified Minimum Estimates; The Limits of Wave Analogies.

Mathematical appendix: Fourteen sections with proofs, explicit constants, exact examples, and limitations.

Research catalogue: Ninety numbered approaches, plus the earlier unnumbered groundwork.

References: Authors, publications, identifiers, and contributions.

Evidence notes: Reproducibility, provenance, and publication status.

Reading conventions. A proved statement, a finite computation, a numerical illustration, and an unresolved hypothesis are labelled separately. Ceiling brackets denote rounding up to the nearest integer; floor brackets denote rounding down. Equation labels from the mathematical appendix are preserved below their equations. Tables in the web sources appear as labeled records in this reading edition.

Scientific status. This report neither proves nor disproves Chowla's square-root conjecture. Lasserre's hierarchy and the Slot–Laurent filtering methods are established prior art. The particular specialization is supplied with proofs, without a claim of priority. The work has not received external peer review or formal proof-assistant verification.

The historical record is supplied for traceability. Reconstructions of approaches 41–50 are explicitly identified, and the missing earlier approach-42 certificates are not presented as recovered evidence. The current curated texts state the corrections that take precedence over older notes.

The Mathematics of Sound and the Search for Negative Peaks

CliDeck Research Notes · Bass Plus · 18 October 2026

Abstract

How far below its average must a collection of synchronized waves fall? This question leads from a simple model of sound to Chowla's cosine problem, a difficult question about sums of distinct integer frequencies. This report brings together a sequence of ninety exploratory approaches, their mathematical checks, and their limitations. It develops exact examples, a spectral interpretation, a counterexample to minimum detection by a natural sampling grid, and an explicit approximation bound whose sufficient polynomial degree depends on the number of frequencies and the requested error, but not on the largest frequency. The approximation method belongs to Lasserre's established pushforward measure hierarchy; its Chebyshev filtering mechanism is closely related to work of Slot and Laurent. We claim neither a solution of Chowla's conjecture nor priority for this general method. The useful outcome is a reproducible account of what can be proved, what finite computations show, and which tempting extensions fail. Eight figures connect the analysis to phase locking, sampling, weighted amplitudes, spectral bands, and finite searches.

1. The question behind the waves

Take a finite set of distinct positive integers A , and let $N = |A|$. We study the real function

$$f_A(x) = \sum_{a \in A} \cos(ax), x \in [0, 2\pi].$$

Every component has amplitude one. Every initial phase is zero. There is one common variable x : the components cannot choose their phases independently. Define the depth of the most negative excursion by

$$K(A) = - \min_{x \in [0, 2\pi]} f_A(x).$$

Chowla's question asks whether there is an absolute constant $c > 0$ such that

$$K(A) \geq c \sqrt{N}$$

for every such set A . The constant would have to work for consecutive frequencies, sparse frequencies, sets with strong arithmetic relations, and arbitrarily large frequency values. There is no upper bound on $\max A$ in the question. Recent papers make substantial progress on weaker polynomial lower bounds, but the sources consulted for this report continue to distinguish that progress from the square-root target. [JMTZ2025] [Bedert2026] [Shankar2026]

The quantifiers are the heart of the problem. Finding a large negative excursion for a particular set is valuable, but proving it for every finite set is a different task. Replacing c by the set-dependent number $c(A) = K(A) / \sqrt{N}$ is an exact description, not a resolution: one must still prove that the infimum of this function stays positive over all sets and all sizes.

For a sound interpretation, set $x = 2\pi \nu_0 t$, where ν_0 is a base frequency in hertz. Then the physical frequencies are $a \nu_0$. The mathematical model describes an ideal harmonic signal with equal amplitudes and synchronized phases. Music and equalized audio usually have unequal amplitudes, changing phases, noise, and finite observation windows. The model helps explain interference and measurement; it does not, by itself, establish a benefit of an audio product.

2. What the coefficient means physically

Use normalized averaging over a full period:

$$E[g] = \frac{1}{2\pi} \int_0^{2\pi} g(x) dx.$$

Orthogonality gives $E[f_A] = 0$ and $E[f_A^2] = N/2$. Thus the root-mean-square amplitude is $\sqrt{N/2}$, while $f_A(0) = N$. All signals in this model have the same RMS amplitude when they have the same number of frequencies. Their negative peaks can nevertheless differ.

For example,

$$K(\{1, 2\}) = \frac{9}{8}, K(\{1, 3\}) = 2.$$

For the first set, put $u = \cos x$ and complete the square:

$$\cos x + \cos 2x = 2 \left(u + \frac{1}{4} \right)^2 - \frac{9}{8}.$$

For the second, both cosines equal -1 at $x = \pi$. The first signal has a shallower trough despite identical RMS amplitude. The dimensionless negative crest factor is

$$CF_-(A) = \frac{K(A)}{\sqrt{N/2}} = \sqrt{2} \frac{K(A)}{\sqrt{N}}.$$

This explains the normalization: c would be a uniform lower bound on a normalized negative peak. It is not an energy quantum, a material constant, or a new universal constant of physics.

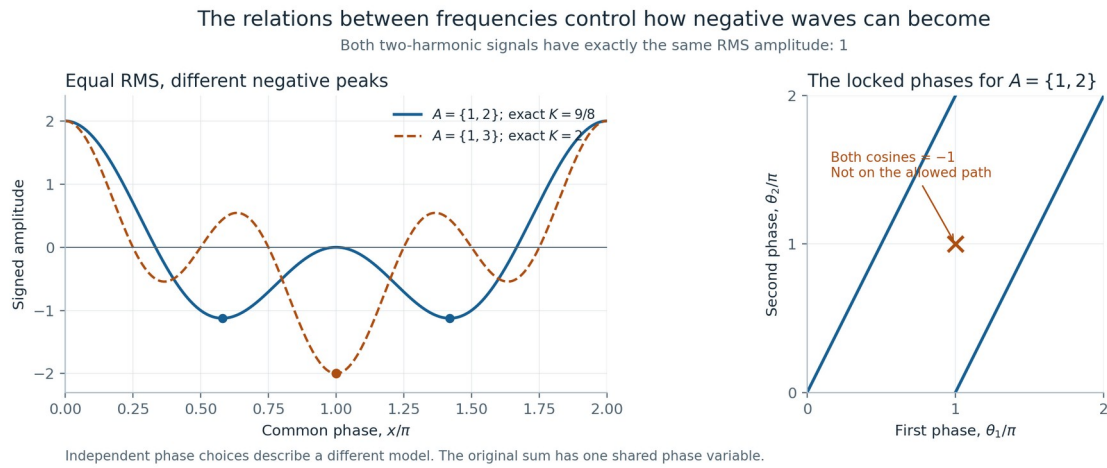


Figure 1. Phase locking changes the attainable trough. The two signals have equal RMS amplitude. For $A = \{1, 2\}$, the hypothetical independent-phase value -2 is inaccessible; the exact common-phase minimum is $-9/8$. The chart keeps the hypothetical model explicitly separate from the actual signal.

If f_A represents acoustic pressure relative to equilibrium, negative values describe rarefaction. They do not mean negative acoustic energy. In the plane progressive-wave approximation, average intensity is proportional to mean-square pressure, with proportionality determined by acoustic impedance. This physical distinction matters whenever a proposed proof replaces a signed wave amplitude by a squared magnitude.
[BakerAcoustics]

3. Why a single common phase is restrictive

The vector $(\cos(a_1 x), \dots, \cos(a_N x))$ follows a one-parameter path. For multiple frequencies, it does not explore the entire cube $[-1, 1]^N$. For $A = \{1, 2\}$, the relation $\cos 2x = 2\cos^2 x - 1$ prevents both coordinates from being -1 at once. Allowing separate phases removes that constraint and changes the problem.

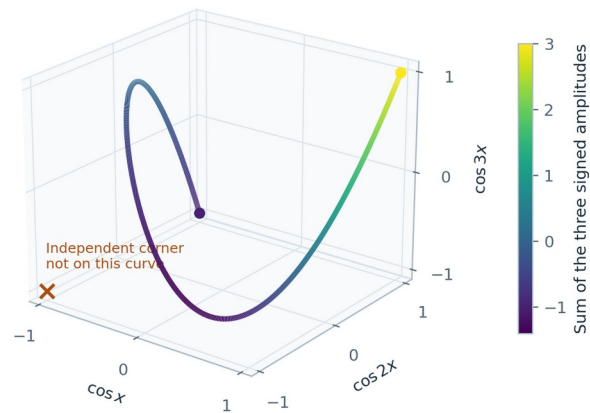
The same point becomes geometric with three frequencies. The path

$$x \mapsto (\cos x, \cos 2x, \cos 3x)$$

is a curve inside a cube. Colouring each point by the sum of its coordinates shows exactly which signal values this shared clock can reach. A rotation of the display changes our viewpoint, not the set of attainable states. This is why a three-dimensional figure is useful here: its geometry represents an actual constraint.

Three wave amplitudes follow one constrained curve

$(\cos x, \cos 2x, \cos 3x)$: colour is their sum, not a probability density



This is an amplitude-coordinate curve, not a path through physical space. A full period traces the curve forward and back.

Figure 2. One clock traces a curve, not a cube. The colour is the actual sum $\cos x + \cos 2x + \cos 3x$. The corner $(-1, -1, -1)$ is not on the curve. Over a full period, the real cosine-coordinate curve is retraced. The figure is a classical phase model, not a visualization of a quantum state.

Multiplying every frequency by the same positive integer d has a different effect. Since $f_{dA}(x) = f_A(dx)$, it compresses the pattern in time while preserving its full-period range:

$$K(dA) = K(A).$$

A trough can become narrower in a local time window without becoming shallower. Any argument that estimates the total negative contribution from one local width must account for the increased number of repeated troughs.

Multiplying all frequencies changes the clock, not the negative depth

$f_{\{5,10\}}(x) = f_{\{1,2\}}(5x)$; both exact negative peaks have depth $9/8$

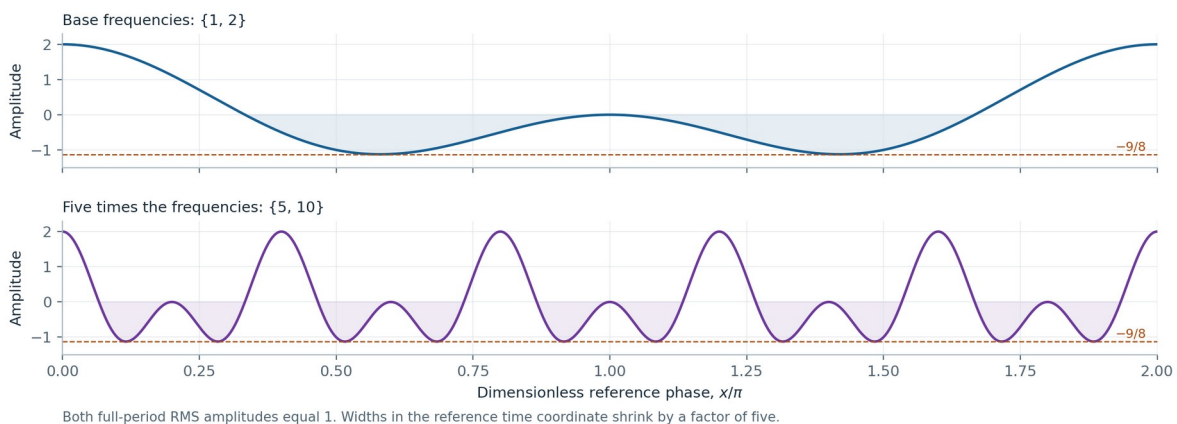


Figure 3. Frequency scale and peak depth are different quantities. The sets $\{1, 2\}$ and $\{5, 10\}$ both have $K = 9/8$. Dilation produces more oscillations within the same displayed interval. It does not improve the normalized negative minimum.

4. A sampling grid can hide the trough

Consider $A = \{1, \dots, N\}$ and let $q = 2N + 1$. On the equally spaced grid $x_j = 2\pi j/q$, a roots-of-unity identity gives

$$f_A(x_0) = N, f_A(x_j) = -\frac{1}{2} (1 \leq j < q).$$

The grid therefore reports a negative depth of only $1/2$. It also reports the correct mean-square value:

$$\frac{1}{q} \sum_{j=0}^{q-1} f_A(x_j)^2 = \frac{N}{2}.$$

Yet at the point between grid locations $x = 3\pi/q$, the Dirichlet-kernel identity yields

$$f_A(3\pi/q) = -\frac{1}{2} - \frac{1}{2 \sin(3\pi/(2q))}.$$

Consequently,

$$K(A) \geq \frac{1}{2} + \frac{q}{3\pi}.$$

The hidden excursion grows linearly with N . This is an exact family of counterexamples to using the smallest raw sample as the true minimum. It is not a failure of trigonometric interpolation: the same grid contains enough information to reconstruct this finite-bandwidth trigonometric polynomial. Reconstructing a signal and simply taking the smallest of its stored samples are different operations.

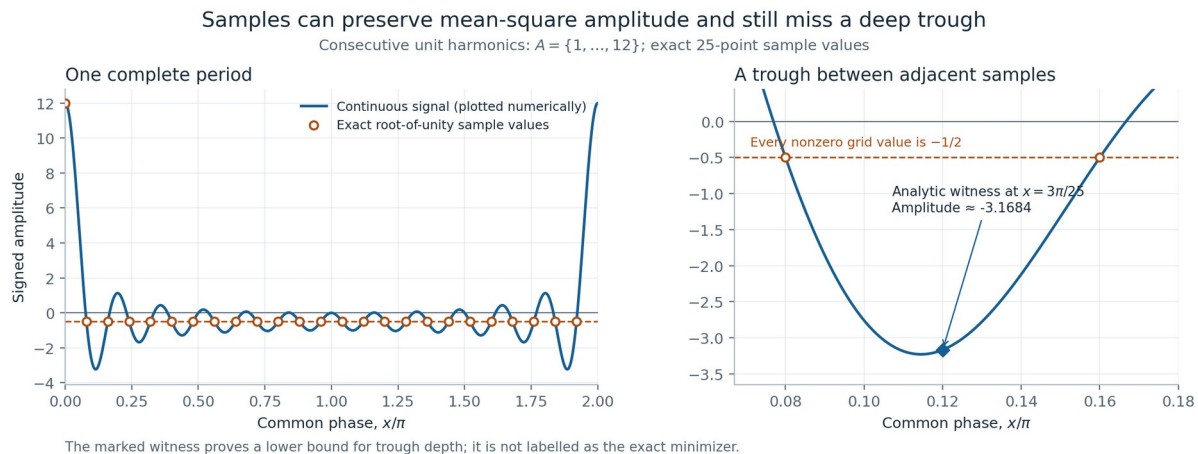


Figure 4. Correct RMS does not certify the minimum. For $N = 12$ and $q = 25$, every nonzero sample is $-1/2$, while the continuous curve passes much lower between samples. The highlighted off-grid point is an exact witness. The drawn smooth curve is an illustration, not a substitute for the identity.

A simple rigorous guard is available for a general set. For a grid with M equally spaced points, let K_M be minus the smallest sample. The derivative bound $|f'_A| \leq \sum a$ gives

$$K_M \leq K(A) \leq K_M + \frac{\pi}{M} \sum_{a \in A} a.$$

A second-derivative argument at a true minimum improves the error bound to $\pi^2 \sum a^2 / (2M^2)$ when that is smaller. A scientific interactive chart should display the certified interval alongside the sampled value. A dense curve by itself does not supply a certificate.

5. The same function as a lattice spectrum

There is an exact way to reinterpret the cosine sum as a particle-like hopping model. Let T be the bilateral shift on square-summable sequences indexed by the integers, and define

$$H_A = \frac{1}{2} \sum_{a \in A} (T^a + T^{-a}).$$

Fourier transformation turns each shift into a complex phase. The operator becomes multiplication by $f_A(x)$, so its spectrum is the full interval

$$\sigma(H_A) = [-K(A), N].$$

This is a mathematical identity. The wave minimum becomes the lower edge of the spectrum of a translation-invariant lattice operator. The model is related to tight-binding descriptions of hopping on a lattice. An infinite lattice can have continuous spectral bands even though its sites are discrete; discrete position labels alone do not quantize all possible energies. [TongAQM]

A finite ring introduces a boundary condition. With q sites and a phase twist ϕ , its eigenvalues are

$$E_j(\phi) = f_A\left(\frac{2\pi j + \phi}{q}\right), 0 \leq j < q.$$

At $\phi = 0$, the ring sees one sampling grid. As ϕ varies through a full turn, the eigenvalue curves sweep the entire continuous phase variable. Taking the minimum over both j and ϕ recovers $-K(A)$ exactly. The boundary twist repairs this finite model's missing phases; it does not create a new universal lower bound. Twisted boundary conditions are an established tool in spectral physics. [Bardyn2014]

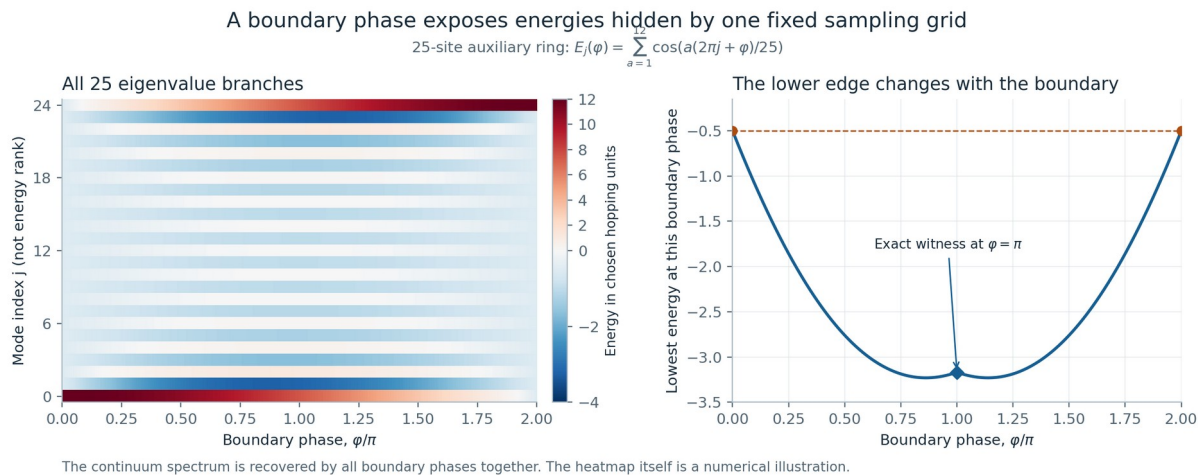


Figure 5. A boundary phase reveals what one finite ring misses. The heatmap shows the actual values $E_j(\phi)$ for $q=25$ and $A=\{1, \dots, 12\}$. The untwisted column has the exceptional two-value spectrum, while other twists expose lower energies. The vertical coordinate labels finite eigenvalue branches rather than an independent sound frequency.

6. A certified way to approximate each minimum

For a real polynomial p of degree at most d , consider the weighted average

$$R(p) = - \frac{E[f_A p(f_A)^2]}{E[p(f_A)^2]}.$$

The denominator is positive for nonzero p because f_A is nonconstant and its range contains an interval. The nonnegative weight $p(f_A)^2$ cannot produce an average below the true minimum. Hence $R(p) \leq K(A)$.

Optimizing over the polynomial gives

$$B_d(A) = \max_{0 \neq p, \deg p \leq d} R(p) \leq K(A).$$

These lower estimates of the depth increase with d . They can be expressed as a generalized eigenvalue problem in moments $\mu_k = E[f_A^k]$. The sign convention matters: the corresponding estimate of $\min f_A$ is an upper bound, while its negative is a lower bound on $K(A)$.

This is the pushforward measure hierarchy developed by Lasserre, specialized to the distribution of a cosine sum. Permitting a sum of polynomial squares instead of one square gives the same optimal quotient: the quotient for a sum is a weighted average of the individual quotients. The construction and its basic convergence are established prior work. [Lasserre2021]

The report provides an explicit sufficient degree tailored to the present family. For $0 < \varepsilon \leq N$, set

$$M = \lceil \frac{8\pi N}{\varepsilon} \rceil.$$

Then the mathematical appendix proves that

$$d \geq \lceil \sqrt{\frac{N}{\varepsilon}} \left[N \log M + \log \left(\frac{16N}{\varepsilon} \right) \right] \rceil \Rightarrow 0 \leq K(A) - B_d(A) \leq \varepsilon.$$

The logarithm is natural. The sufficient degree depends on N and the requested absolute error, but not on $\max A$. The proof first shows that a set of phases close to a minimum has a uniform positive measure, using a partition of the phase torus. A Chebyshev polynomial then gives that region a sufficiently large relative weight. The underlying filtering architecture is closely related to the needle-polynomial analyses of Slot and Laurent. The frequency-independent specialization and its constants are proved here for review; their priority has not been established. [SlotLaurent2021] [SlotLaurent2022] [NISTDLMF]

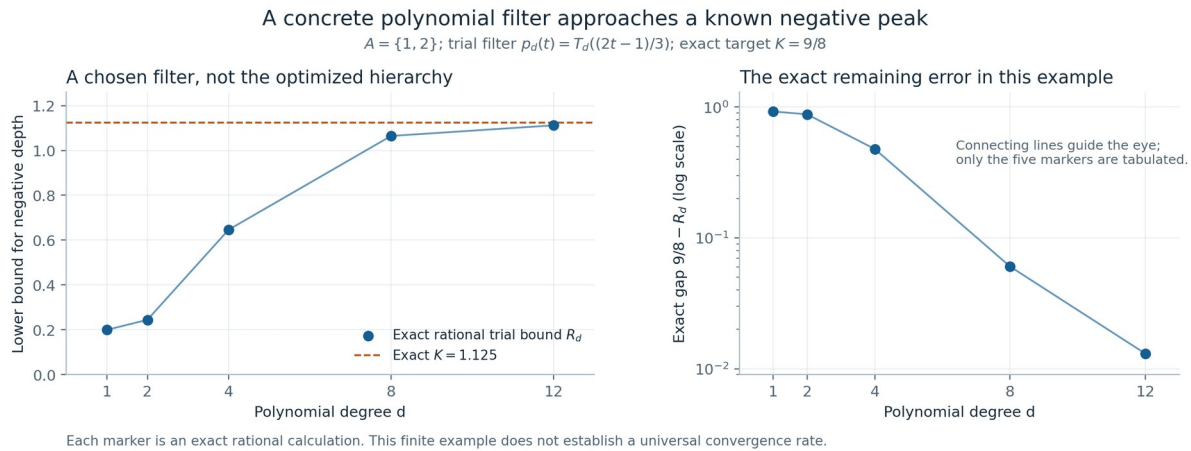


Figure 6. A valid approximation can improve without solving the conjecture. For $A = \{1, 2\}$, an explicitly chosen Chebyshev trial polynomial gives exact rational values at the highlighted degrees, approaching the known depth $9/8$. These are values of a particular trial quotient R_d , not the optimized hierarchy B_d . The comparison line is an exact minimum in this special example.

This result answers an approximation question for each input. It does not answer the extremal question over all inputs. The degree may be large, exact moments may be expensive to compute, and a generalized eigenvalue calculated in floating point requires an error certificate before being called exact. A finite sufficient degree is not a claim of practical polynomial runtime in the input length.

7. Negative results that narrow the search

Some failed extensions have clean mathematical explanations. Their value is to prevent the same obstruction from being mistaken for progress again.

Unequal amplitudes change the theorem

For weights $w_a = 1 - a/(N + 1)$ on $a = 1, \dots, N$, the Fejér-kernel identity gives

$$\sum_{a=1}^N w_a \cos(ax) = \frac{F_N(x) - 1}{2} \geq -\frac{1}{2},$$

and equality is attained. Meanwhile $\sum w_a^2 = N(2N + 1)/(6(N + 1))$ grows linearly in N . Therefore the negative depth divided by the coefficient norm tends to zero. Multiplying by $N + 1$ gives positive integer weights and the same failure after normalization. Making amplitudes integer-valued does not restore the unit-amplitude conjecture.

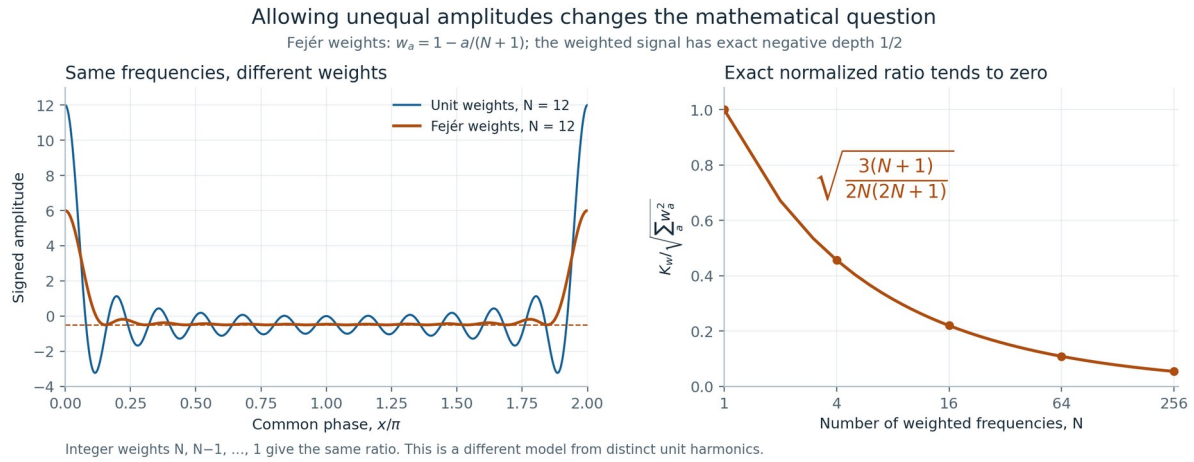


Figure 7. A sharp positive pulse can coexist with a fixed negative floor. Fejér weights give an exact depth of $1/2$ while the coefficient norm grows. The companion normalization plot explains why this is a counterexample to a weighted extension, not to Chowla's original equal-amplitude problem.

A physically inspired weight can emphasize too little

The logistic or Fermi–Dirac-inspired weight $w_\beta(x) = 1 / (1 + e^{\beta f_A(x)})$, with $\beta \geq 0$ and threshold fixed at zero, defines a valid lower estimate

$$B_\beta = - \frac{E[f_A w_\beta]}{E[w_\beta]}.$$

For consecutive frequencies, the appendix proves $B_\beta \leq 20 + 10 \log(2N + 1)$ for every β . Choosing a larger inverse-temperature parameter cannot turn this particular family into a square-root certificate on every input. An adjustable threshold would be a different family and is not ruled out by this statement.

A promising finite minimum remains finite evidence

Another nonnegative weight is the Riesz product $\prod_{a \in A} (1 - \cos ax)$. Its associated quotient can be calculated exactly from integer Laurent-polynomial coefficients. The recorded exhaustive calculation checks all 262,143 nonempty subsets of $\{1, \dots, 18\}$. On that finite domain, the smallest normalized quotient is $7/20$, attained at $A = \{1, 2, 3, 4, 5, 7, 9, 11, 13\}$.

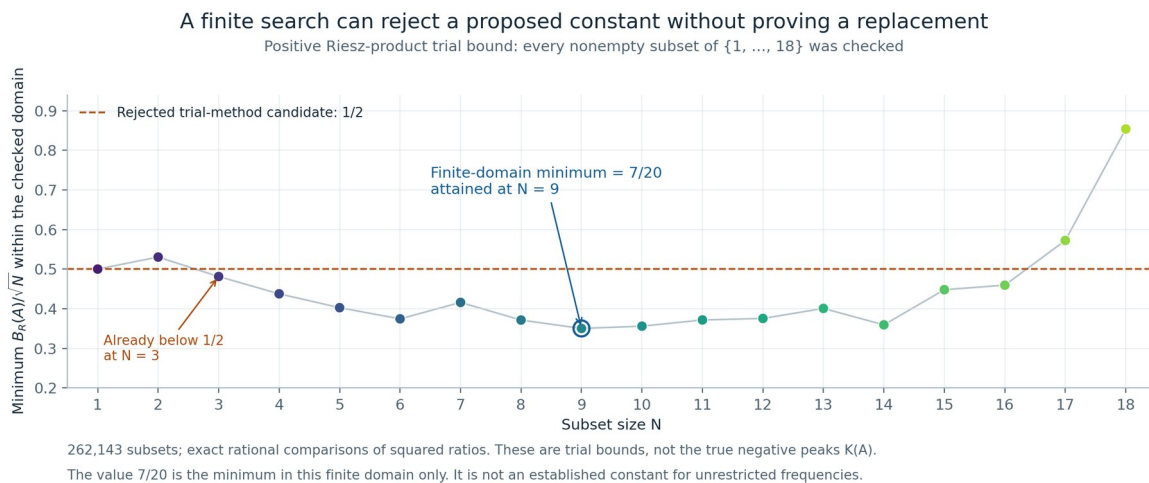


Figure 8. A finite search suggests questions; it does not settle all frequencies. The plotted minima are grouped by set size within the fixed universe $\{1, \dots, 18\}$. The value $7/20$ is an exact finite-domain result. The rejected candidate $1/2$ fails already for $A = \{1, 2, 3\}$. No universal lower bound for this Riesz quotient is claimed.

8. What the quantum analogy contributes

The wave-to-lattice correspondence is useful because it is exact and preserves the quantity being studied. Other quantum analogies need an equally explicit dictionary. For a harmonic oscillator, a number state has zero mean displacement even when it has nonzero occupation. A coherent state can have a sinusoidal mean displacement, while its mean occupation need not be an integer. The integer spectrum of the number operator does not force the Fourier coefficients of a classical waveform to be integers. Squeezing changes quadrature variances; it need not change the mean waveform. [Cappellaro2012] [Shapiro6453]

Experiments can control individual mechanical quanta, so phonons are physical rather than merely metaphorical. That does not transfer a quantum measurement theorem to a signed classical cosine sum without proof. [OConnell2010] In this investigation, the productive question was not whether an analogy sounded surprising, but whether it preserved amplitudes, phases, averaging, and the minimum. The catalogue records where those requirements succeeded and where they failed.

9. Evidence, provenance, and the remaining problem

The companion mathematical appendix supplies full proofs and separates identities, inequalities, and finite examples. The research catalogue indexes ninety numbered approaches and an earlier unnumbered groundwork stage. The archive includes the preserved source notes, code, result files, and three sealed historical archives. It also identifies corrections that supersede earlier statements.

A selected reproduction of the early groundwork checked all 4,194,303 nonempty subsets of $\{1, \dots, 22\}$ with two different enumeration procedures. Both found the same unique minimizer of $K(A)/\sqrt{|A|}$ within that finite universe: $A = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 21\}$. Exact certificates put its depth strictly between 2.4548327530 and 2.4548327531. This concerns the actual continuous minimum; the separate Riesz calculation above concerns a particular lower-estimate method. Neither finite result covers arbitrarily large frequencies. The reproduction report and its machine-readable outputs are included in the archive.

There is a material provenance limitation: approaches 41–50 were reconstructed from surviving descriptions rather than recovered as original bytes. In particular, the earlier claim involving 1,980 finite certificates for approach 42 cannot be represented as a reproduced certificate set; those files are missing. A separately available tail argument has a different verification status. These distinctions are documented in the catalogue and provenance notes.

The checks in this project consist of mathematical derivations, exact arithmetic programs, independent calculations within this workflow, and reproducible numerical illustrations. They are not external peer review or formal proof-assistant verification. ChatGPT assisted the writing, derivations, literature comparison, and code preparation. No claim of independent verification by OpenAI is made.

What remains is precise: prove a positive universal lower bound for $K(A)/\sqrt{|A|}$, or produce a valid sequence of unit-amplitude, zero-phase, distinct positive integer frequency sets for which that ratio tends to zero. None of the weighted examples, finite grids, independent-phase models, or finite searches in this report accomplishes either task. The present contribution is a checked collection of tools and obstructions, with a clear route for readers to inspect and extend the work.

Further reading and supporting material

The accompanying pages are **Sampling and Hidden Peaks**, **Waves and Lattice Particles**, **Certified Minimum Estimates**, **The Limits of Wave Analogies**, **Research Catalogue of Ninety Approaches**, and the **Mathematical Appendix**. Full bibliographic entries appear on the **References** page. The downloadable research edition includes all of these texts and the reproducibility materials.

Sampling and Hidden Peaks

A waveform plot is a collection of evaluations joined by a drawing rule. Unless an error estimate is supplied, the lowest visible point is only an estimate of the true trough. A particularly instructive example shows that even an exactly correct RMS calculation cannot repair this gap.

An exact sampling trap

Take $f_N(x) = \sum_{a=1}^N \cos(ax)$, and sample it at the $q = 2N + 1$ points $x_j = 2\pi j / q$. The sum of all q complex roots of unity is zero at every nontrivial root. Pairing conjugate terms gives

$$1 + 2 \sum_{a=1}^N \cos(2\pi a j / q) = 0 \quad (1 \leq j < q).$$

Thus the samples are N once and $-1/2$ at every other point. Their mean is zero and their mean square is

$$\frac{N^2 + 2N(1/4)}{2N + 1} = \frac{N}{2}.$$

These are the true continuous mean and mean square. The sample minimum, however, is not the true minimum.

The Dirichlet identity is

$$1 + 2f_N(x) = \frac{\sin((N + 1/2)x)}{\sin(x/2)},$$

with the values at removable singularities defined by continuity. At $x = 3\pi / q$, its numerator is -1 , so

$$-f_N(3\pi / q) = \frac{1}{2} + \frac{1}{2 \sin(3\pi / (2q))} \geq \frac{1}{2} + \frac{q}{3\pi}.$$

The final inequality uses $\sin u \leq u$ for positive u . For increasing N , the missed depth becomes arbitrarily large even though every nonzero sample remains fixed at $-1/2$.

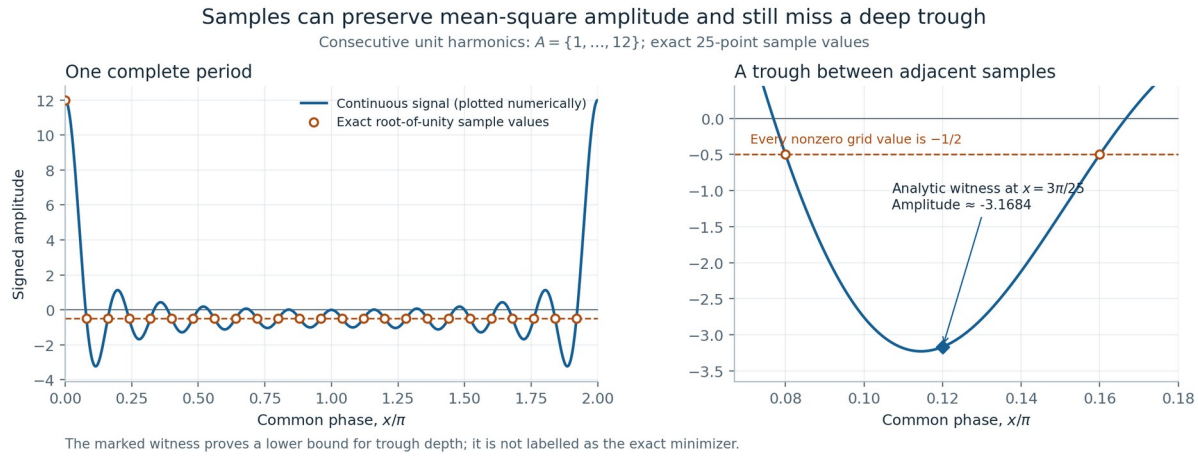


Figure. The default demonstration uses $N = 12$ and $q = 25$. The off-grid marker is an exact witness, not an assertion that its location is the exact global minimizer. The continuous line should be labelled as a numerical drawing of the known analytic function.

Why this does not contradict signal reconstruction

This function has known bandwidth and is a trigonometric polynomial of degree N . Its $2N + 1$ samples determine all of its coefficients. One can reconstruct the polynomial and then minimize it with a separate, controlled procedure. The failure occurs when the smallest recorded sample is treated as the minimum of the reconstructed continuous signal. The information is present, but the operation performed on it is insufficient.

That distinction is useful for a spectrum display or audio-analysis interface. A responsive plot can use a finite grid, but its text should not imply more accuracy than the computation supplies. The same caution applies when a numerical experiment searches many frequency sets: a small observed value of K could simply be a trough the grid did not see.

A rigorous interval around a sampled estimate

For a general set A , use M equally spaced points and define $K_M = -\min_j f_A(2\pi j/M)$. Every point of the circle is within π/M of a grid point. Since

$$|f'_A(x)| \leq S_1, S_1 = \sum_{a \in A} a,$$

the true depth lies in $[K_M, K_M + \pi S_1/M]$. At a true minimizer the first derivative vanishes. Taylor's theorem and $|f''_A| \leq S_2$, where $S_2 = \sum a^2$, also give

$$K(A) \in \left[K_M, K_M + \min \left(\frac{\pi S_1}{M}, \frac{\pi^2 S_2}{2M^2} \right) \right].$$

These inequalities are exact in real arithmetic. A software implementation must add a numerical evaluation tolerance if it claims a rigorously certified floating-point result. Merely printing many decimal places does not provide that tolerance.

What the interactive version should show

The useful interaction is to move the sample grid while retaining the continuous function and the off-grid witness. A second control changes the number of samples. The readout should distinguish **sampled depth**, **analytic witness**, and **provable discretization error**. When the witness is used, the lower endpoint can be increased to the larger of the sampled value and the witness depth.

The animation should begin only when the reader chooses Play. A paused slider is equally informative and easier to inspect. All states need keyboard controls and a static fallback. There is no reason to request microphone access or play sound: this example explains the geometry of measurement using an ideal mathematical signal.

For a related spectral version, see **Waves and Lattice Particles**. The full derivation, including the twisted finite ring, appears in the **Mathematical Appendix**.

Waves and Lattice Particles

An analogy becomes mathematically useful when it translates the quantity we care about without changing the question. For a cosine sum, a lattice model does exactly that. Its connection to quantum mechanics is informative, but the identity can be understood using shifts and Fourier series alone.

A hopping operator with the same dispersion

Let $\ell^2(\mathbb{Z})$ be the space of square-summable sequences on the integer lattice. Write $T e_n = e_{n+1}$ for the shift of a basis vector. For a finite frequency set A , define

$$H_A = \frac{1}{2} \sum_{a \in A} (T^a + T^{-a}).$$

The operator permits a hop by each distance a in both directions. It is self-adjoint because every forward shift is paired with its adjoint. Under the unitary Fourier transform $e_n \mapsto e^{inx}$, it acts as multiplication by

$$f_A(x) = \sum_{a \in A} \cos(ax).$$

The spectrum of a multiplication operator is its essential range. Here continuity, together with the connected circle and full support of uniform measure, makes this the interval $[-K(A), N]$. The lowest spectral value and the most negative wave value are therefore the same number.

This resembles the dispersion relation of a tight-binding model. The analogy concerns a linear operator and its spectrum; it does not assert that a classical audio wave is a stream of localized particles. Standard lattice physics also distinguishes a finite system's eigenvalues from the continuous bands of an infinite periodic system. [TongAQM]

A finite ring sees selected phases

On a ring of q sites, introduce a phase ϕ at the closing boundary. Choose the shift so that $T_\phi^q = e^{i\phi} I$. Its eigenphases are $(2\pi j + \phi)/q$, and hence

$$E_j(\phi) = f_A((2\pi j + \phi)/q), j = 0, \dots, q-1.$$

The untwisted ring samples one grid. Sweeping $0 \leq \phi \leq 2\pi$ and all j covers a whole period. Consequently,

$$\min_{\phi, j} E_j(\phi) = -K(A).$$

For $A = \{1, \dots, N\}$ and $q = 2N + 1$, the untwisted operator is $(J - I)/2$, where J is the all-ones matrix. Its eigenvalues are N once and $-1/2$ with multiplicity $2N$. That deceptively shallow lower value is exactly the sampling trap from the companion article. Boundary twisting restores the missing phases. Twisted boundary conditions have an established role in spectral and topological analysis. [Bardyn2014]

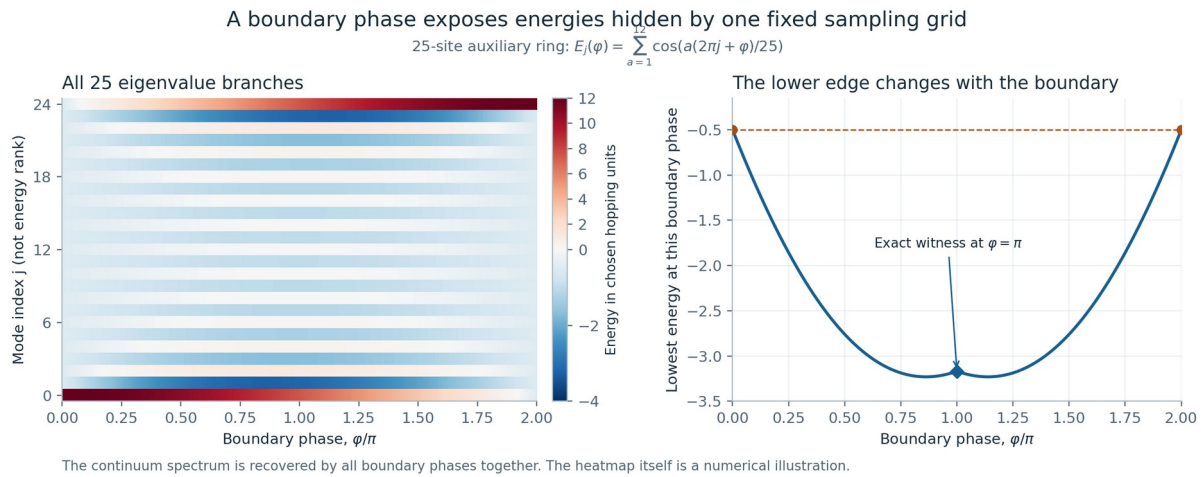


Figure. Each column corresponds to one boundary phase. The colours encode eigenvalue values computed from the explicit cosine formula. An interactive crosshair can show the associated continuous phase χ on the waveform. Colours represent signed spectral values, not probability or acoustic energy.

Open boundaries give another approximation

Compress the infinite operator to L consecutive sites and call the result $H_{A,L}$. Let $B_L = -\lambda_{\min}(H_{A,L})$. The variational principle gives $B_L \leq K(A)$, and enlarging the box makes these lower estimates nondecreasing. A truncated plane wave supplies the explicit bound

$$0 \leq K(A) - B_L \leq \sum_{a \in A} \min\left(\frac{a}{L}, 1\right).$$

This is a useful convergence statement for a fixed set. It is not uniform in unrestricted frequencies at a fixed box size: if every hop is at least L , the compression can have no nonzero entries at all. The finite model then loses the dynamics it was meant to represent. Boundary choices are part of the model, not a cosmetic implementation detail.

Symmetry explains an extreme case

If every frequency divided by their common gcd is odd, a common phase makes every cosine -1 , and $K(A) = N$. The normalized lattice graph is then bipartite: a hop flips the parity of the site. Its spectrum is symmetric about zero. Conversely, reaching $-N$ requires all cosines to equal -1 , which forces exactly this parity condition. The appendix proves both directions.

Odd relations between hop lengths obstruct simultaneous anti-alignment. They can provide an upper restriction on how close the minimum can come to $-N$. Such an upper bound must not be confused with the lower bound needed for Chowla's conjecture.

The spectral translation is productive because these statements preserve the original minimum. It supplies variational tools, boundary diagnostics, and exact examples. A universal square-root lower bound still needs a further argument about all allowable frequency sets.

Certified Minimum Estimates

Computing a very low value of a waveform proves that such a value occurs. It does not prove that no lower value occurs elsewhere. A useful approximation framework makes the direction of every estimate explicit and eventually places the true minimum inside a controlled interval.

Turning nonnegative weights into bounds

For $f = f_A$ and any nonzero real polynomial p , the function $p(f)^2$ is a nonnegative weight. Its weighted average of f cannot be less than $\min f$. Therefore

$$R(p) = -\frac{E[f p(f)^2]}{E[p(f)^2]} \leq K(A).$$

Optimizing R over degree at most d defines $B_d(A)$. Increasing the degree enlarges the choice of weights, so B_d is nondecreasing. This construction is an instance of Lasserre's pushforward measure hierarchy. Its general mechanism and convergence are established results, not discoveries claimed by this report. [Lasserre2021]

If $p(t) = \sum_{i=0}^d c_i t^i$ and $\mu_k = E[f^k]$, the two quadratic forms use the matrices

$$H_0 = (\mu_{i+j})_{0 \leq i, j \leq d}, H_1 = (\mu_{i+j+1})_{0 \leq i, j \leq d}.$$

Thus B_d is minus the least generalized eigenvalue of (H_1, H_0) . For a cosine sum, moments can be computed from coefficients of a Laurent polynomial: integration selects its constant coefficient. Exact integer and rational arithmetic can verify a specified trial quotient without relying on numerical quadrature.

Why Chebyshev polynomials help

Chebyshev polynomials remain bounded on $[-1, 1]$ and grow rapidly outside that interval. After rescaling the signal range, this lets a polynomial give much greater weight to values near the minimum than to values farther away. That contrast, together with a lower bound on the measure of the near-minimum region, controls the weighted average. This is closely related to the needle-polynomial convergence analyses of Slot and Laurent. [SlotLaurent2021] [SlotLaurent2022] [NISTDLMF]

The appendix proves a version uniform in the largest frequency. For $0 < \varepsilon \leq N$, define $M = \lceil 8\pi N / \varepsilon \rceil$. The sufficient degree is

$$d \geq \lceil \sqrt{\frac{N}{\varepsilon}} [N \log M + \log(16N / \varepsilon)] \rceil.$$

At this degree, $0 \leq K(A) - B_d(A) \leq \varepsilon$ for every set of N distinct positive integer frequencies. The proof uses the geometry of all N phases together rather than a derivative bound involving the largest frequency. The formula is a sufficient bound, not an optimal one. Its constants are supplied for verification; priority for this specialization has not been established.

The proof also has a certificate using positive integers. One can choose M_0, s, q, p with $M_0 \varepsilon \geq 32 N$, $s^2 \varepsilon \geq N$, $2^q \geq M_0$, and $2^p \varepsilon \geq 16 N$. Then $d = s(Nq + p)$ is sufficient. Requiring $s > 0$ is essential. This form avoids trusting a floating-point logarithm when checking the sufficient degree for rational input.

A concrete exact illustration

For $A = \{1, 2\}$, the true depth is $9/8$. Choose the specific trial polynomial $T_d((2t - 1)/3)$, where T_d is the Chebyshev polynomial of degree d . Exact moment arithmetic gives the following values of the associated quotient.

Degree d : 1 Exact trial value R_d : $1/5$

Degree d : 2 Exact trial value R_d : $12/49$

Degree d : 4 Exact trial value R_d : $4656/7201$

Degree d : 8 Exact trial value R_d : $601297088/564810049$

Degree d : 12 Exact trial value R_d : $84674399485872/76148427919201$

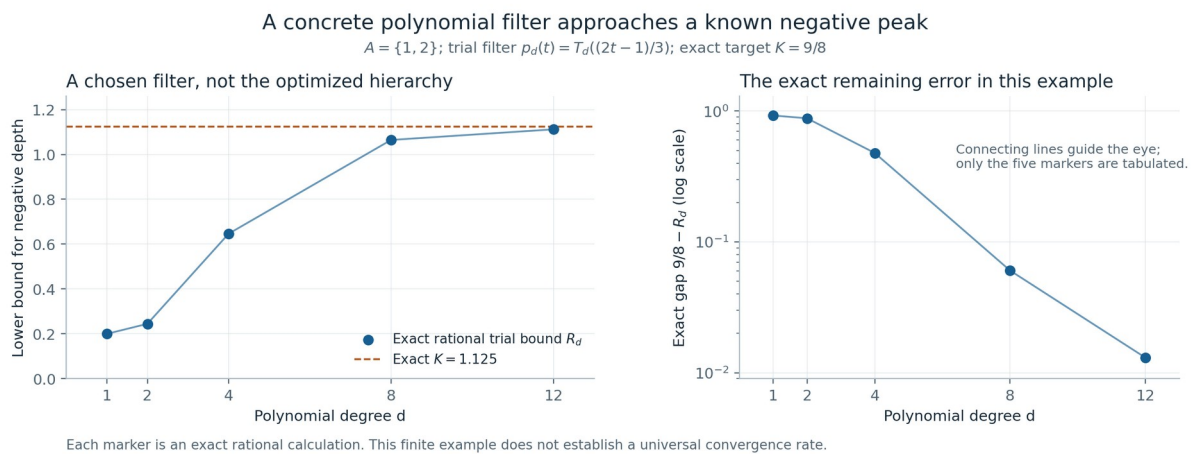


Figure. The highlighted rational values are certified trial values. The optimized quantity B_d can only be larger. The graph must not relabel this particular sequence as the optimized hierarchy or suggest that monotonicity of all trial choices is guaranteed. The monotonicity theorem concerns B_d .

What the result does and does not compute

The uniform degree bound removes dependence on frequency scale from this particular approximation guarantee. It does not make moment generation free. Large degrees can require many large integers, and a monomial moment matrix can be ill-conditioned in floating point. A plotted generalized eigenvalue without a certified numerical error is still an estimate.

There is also an unavoidable distinction between approximation and an extremal theorem. Approximating $K(A)$ for each input does not prove that the resulting values are all at least $c\sqrt{N}$. That further inequality is exactly the unresolved target of Chowla's problem. The approximation theorem is useful because it states clearly what accuracy can be obtained and under which assumptions, not because it bypasses that target.

The Limits of Wave Analogies

The research explored ideas from physics, biology, social dynamics, and optimization. Their usefulness depended on an exact question: which mathematical quantity survives the translation? An attractive analogy can generate a hypothesis, but its conclusion needs a proof in the original model.

Negative pressure is not negative energy

A cosine sum is signed. Its negative trough is the object of Chowla's problem. Acoustic pressure relative to equilibrium can also be negative, describing rarefaction. Average acoustic intensity, under the usual plane progressive-wave assumptions, depends on mean-square pressure and acoustic impedance. Squaring a waveform discards its sign, so an energy statement does not automatically constrain a negative pressure excursion. [BakerAcoustics]

A similar issue arises with a complex amplitude. Let $S_A(x) = \sum_{a \in A} e^{iax}$. The sets $A = \{1, 2\}$ and $A = \{2, 3\}$ have exactly the same $|S_A(x)|^2 = 2 + 2 \cos x$, because one complex sum is obtained from the other by a unit-modulus phase factor. Their real parts have different negative minima. This example concerns the squared magnitude of a complex amplitude; it does not assert equality of the squares of their real waveforms. That qualification prevents a misleading physical interpretation.

Equal energy does not fix a signed trough

For distinct unit-amplitude cosine frequencies, the full-period mean square is always $N/2$. The two sets $\{1, 2\}$ and $\{1, 3\}$ therefore have the same RMS amplitude while their negative depths are $9/8$ and 2 . Any proposed formula using only this energy statistic must miss information about arithmetic phase constraints.

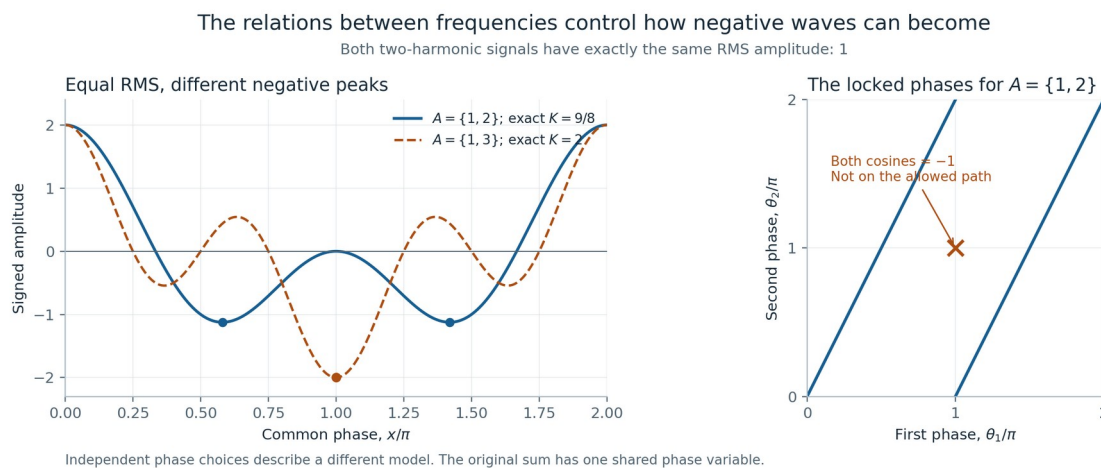


Figure. The comparison separates the average squared amplitude from the attainable negative peak. It also marks the independently adjustable phase value as belonging to a different model.

Integer amplitudes are not enough

The user-inspired question about quantization led to a precise test: would restricting coefficients to positive integers restore a square-root lower bound normalized by coefficient energy? The answer is no. Start with the Fejér weights $w_a = 1 - a/(N+1)$. The nonnegative Fejér kernel gives a weighted negative depth of exactly $1/2$. Scaling to integer weights $v_a = N+1 - a$ gives

$$K_v = \frac{N+1}{2}, \sum_{a=1}^N v_a^2 = \frac{N(N+1)(2N+1)}{6}.$$

Consequently $K_v / (\sum v_a^2)^{1/2} \rightarrow 0$. This is a rigorous obstruction to the weighted extension. It leaves the original unit-amplitude problem untouched because the coefficients in that problem must all equal one.

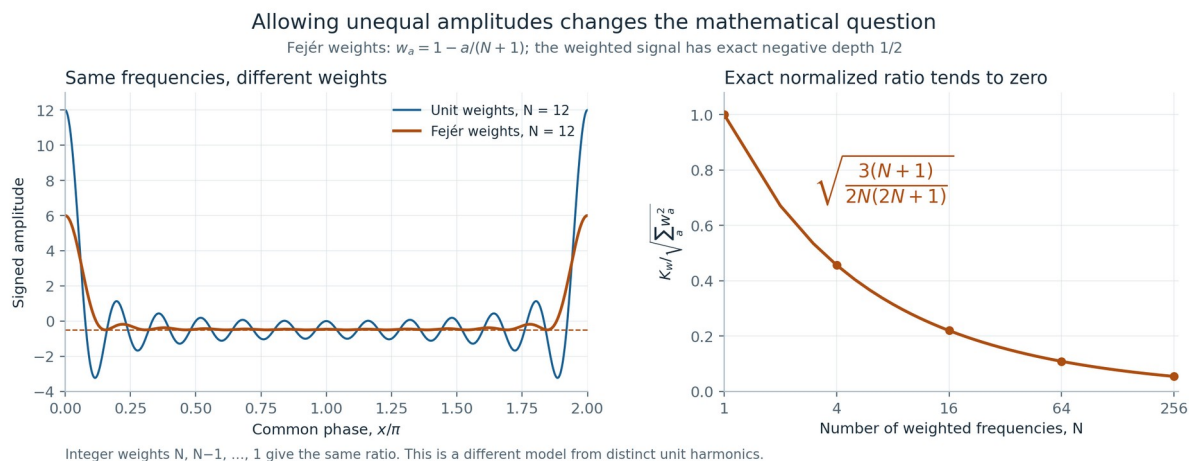


Figure. The pulse shape and the normalization curve explain the counterexample together. The tall positive pulse is compensated by a shallow negative floor over a broad region.

What phonons actually quantize

For a harmonic oscillator, the number operator has integer eigenvalues and the energy levels are $\hbar \omega (n + 1/2)$. That is a statement about measurement outcomes in a quantum model. In a number state, the mean displacement is zero. For a coherent state with complex parameter α , the mean occupation is $|\alpha|^2$, which need not be an integer, while the mean displacement oscillates sinusoidally. A classical Fourier coefficient can be associated with a coherent mean without becoming an integer. [Cappellaro2012] [Shapiro6453]

Squeezing further illustrates the distinction: it changes quadrature variances while a chosen displacement can preserve the mean waveform. A bound on fluctuations and a bound on the signed mean are different statements. Experiments that control a single mechanical quantum establish the physical relevance of phonons, but do not supply a theorem about every classical cosine sum. [OConnell2010]

The exact lattice operator in **Waves and Lattice Particles** is more productive for this particular problem because it preserves the minimum itself. Even there, the spectral correspondence does not prove the desired universal lower bound.

Biology and social dynamics as search tools

Ideas such as selection, competition, and consensus can suggest a nonnegative weighting scheme or an optimization process. Once translated, the method has to specify its objective, normalization, and domain. A replicator-like rule may concentrate weight on low values, but convergence to a minimum says nothing yet about how negative that minimum must be for every frequency set. A game may become easier when its adversary is restricted to a finite frequency range; that restricted value is not automatically the value of the unrestricted game.

The **Research Catalogue of Ninety Approaches** records these translations and their actual outcomes. Some produce exact identities, some give approximations or useful counterexamples, and some remain unproved hypotheses. Treating those statuses separately makes unusual approaches scientifically useful: it lets another reader reproduce the successful step, understand the failed implication, and formulate a different next test.

Mathematical Appendix: Certificates, Spectral Models, and Limitations for Cosine Sums

Companion to the Bass Plus “Mathematics of Sound” research report.

Version: 18 October 2026.

Scope: complete statements and proofs for the results promoted in the accompanying articles.

This appendix studies a precise model of synchronized, equally weighted harmonics. It does not prove or disprove Chowla’s cosine conjecture. Its purpose is to make the report independently checkable: every bound states its domain, every computational observation identifies its finite search space, and every change of model is explicit.

The moment hierarchy below is a specialization of **Lasserre’s univariate pushforward-measure hierarchy** [Lasserre2021]. Polynomial concentration using Chebyshev or needle polynomials is established prior art, including the analyses of Slot and Laurent [SlotLaurent2021]; [SlotLaurent2022]. We therefore make no claim to have invented either mechanism. The maximum-frequency-independent degree estimate proved here is a specialized bound obtained during this investigation; its novelty relative to all existing literature has not been established. The proofs below stand independently of a novelty claim.

Contents

- A. Model, normalization, and the constant
- B. Attainment at a fixed number of frequencies
- C. Exact moments and the polynomial hierarchy
- D. A phase-mass lemma independent of the highest frequency
- E. Uniform approximation using a Chebyshev filter
- F. Consecutive frequencies and low-degree certificates
- G. The lattice operator and a finite open box
- H. A discrete grid can miss the minimum
- I. Why integer amplitudes do not preserve the conjecture
- J. A positive Riesz-product certificate
- K. Negative moments, quantiles, and a saturating detector
- L. Gibbs weights and an entropy budget
- M. Frequency scaling and finite observation time
- N. What has and has not been established

A. Model, normalization, and the constant

Let

$$A = \{a_1, \dots, a_N\} \subset \mathbb{Z}_{>0}, N \geq 1,$$

where the frequencies are distinct. Throughout the original problem,

$$f_A(x) = \sum_{a \in A} \cos(ax), K(A) = -\min_{x \in \mathbb{R}} f_A(x).$$

Equation (A.1)

The variable belongs to the circle $T = \mathbb{R} / (2\pi \mathbb{Z})$. There is one common variable, not an independently adjustable phase for each frequency. All amplitudes equal one, and all initial phases equal zero. Write

$$Eu = \int_T u(x) d\mu(x), d\mu(x) = \frac{dx}{2\pi}.$$

The measure has total mass one. Consequently, different positive integer harmonics are orthogonal and

$$Ef_A = 0, Ef_A^2 = \frac{N}{2}, -N \leq f_A(x) \leq N, f_A(0) = N.$$

Equation (A.2)

For completeness, writing each cosine as $(e^{iax} + e^{-iax})/2$ reduces these assertions to $Ee^{ikx} = 0$ for every nonzero integer k , and to one when $k = 0$. Continuity makes the minimum in (A.1) attainable. Its value is negative, since a nonconstant continuous function of mean zero cannot be nonnegative everywhere.

Proposition A.1 — An elementary uniform bound. For every admissible A ,

$$1 \leq K(A) \leq N.$$

Equation (A.3)

Proof. Only the lower bound requires an argument. Let $D = \max A$, and take the D points

$$x_j = \frac{(2j+1)\pi}{D}, 0 \leq j < D.$$

For $1 \leq a < D$, the average of e^{iax_j} equals

$$\frac{e^{ia\pi/D}}{D} \sum_{j=0}^{D-1} e^{2\pi i a j/D} = 0.$$

For $a = D$, every $\cos(Dx_j) = -1$. The average of f_A over these points is therefore exactly -1 , so at least one point has value at most -1 . The upper bound follows from $f_A \geq -N$. $\square$

The coefficient attached to an individual set is

$$c(A) = \frac{K(A)}{\sqrt{N}}.$$

A physically scaled pressure perturbation might be represented by

$$p(t) = P_0 f_A(\omega_0 t), P_0 > 0, \omega_0 > 0.$$

Over a full period, $p_{rms} = P_0 \sqrt{N/2}$, while its negative peak magnitude is $P_0 K(A)$. Thus its negative peak-to-RMS ratio is exactly $\sqrt{2} c(A)$. The ratio is dimensionless and unchanged by either common scale. This interpretation concerns the shape of the waveform; it does not identify $c(A)$ with an energy quantum or a material constant.

To formulate the open question, define

$$\kappa_N = \inf_{|A|=N} K(A), c_N = \frac{\kappa_N}{\sqrt{N}}.$$

Equation (A.4)

Chowla's assertion is the existence of an absolute $c > 0$ for which

$$K(A) \geq c \sqrt{|A|} \text{ for every admissible } A.$$

Equation (A.5)

Equivalently, $\inf_{N \geq 1} c_N > 0$. It is also equivalent to $\liminf_{N \rightarrow \infty} c_N > 0$: a positive eventual lower bound can be combined with $c_N \geq 1/\sqrt{N}$ for the finitely many remaining sizes. In contrast, positivity of each individual c_N follows already from (A.3) and does not settle (A.5).

A disproof would require actual admissible sets A_j with $|A_j| \rightarrow \infty$ and $K(A_j)/\sqrt{|A_j|} \rightarrow 0$. A lower-bound formula that becomes small does not supply such a sequence: the formula may merely fail to detect a much deeper minimum.

B. Attainment at a fixed number of frequencies

The infimum in (A.4) is taken over infinitely many frequency sets. Nevertheless it is attained. This fact does not give an effective bound on the largest frequency of an exact minimizer.

Theorem B.1 — Fixed-size attainment. For each integer $N \geq 1$, some admissible set A of size N satisfies $K(A) = \kappa_N$.

Proof. Choose admissible ordered tuples $a^{(j)} = (a_1^{(j)}, \dots, a_N^{(j)})$ with $K(A_j) \rightarrow \kappa_N$, and let

$$H_j = \{a^{(j)} x \bmod 2\pi : x \in T\} \subset T^N.$$

Each H_j is a compact circle subgroup. The nonempty compact subsets of a compact metric space are compact for the Hausdorff metric; one proof chooses convergent membership patterns on successively finer finite nets and takes a diagonal subsequence. Pass to a subsequence with Hausdorff limit H . This limit is a subgroup: approximating any $u, v \in H$ by $u_j, v_j \in H_j$ shows that $u - v$, the limit of $u_j - v_j$, belongs to H .

Define the integer annihilator

$$L(H) = \{k \in \mathbb{Z}^N : e^{ik \cdot \theta} = 1 \text{ for every } \theta \in H\}.$$

We use two elementary facts about this group. First, every subgroup of $\mathbb{Z}^N$ is finitely generated. An induction proves this: project onto the first coordinate, whose image is either zero or $d\mathbb{Z}$; choose one lift of d in the latter case, and apply the induction to the kernel. Second,

$$H = \{\theta : e^{ik \cdot \theta} = 1 \text{ for every } k \in L(H)\}.$$

Equation (B.1)

Here is a direct justification of the separation statement (B.1). With a translation-invariant metric on the torus, the continuous function $r(\theta) = \text{dist}(\theta, H)$ is invariant under translations by H . Its Fourier coefficient at k vanishes unless $k \in L(H)$; translate the defining integral by any $h \in H$. Product Fejér convolutions approximate r uniformly, and contain only Fourier coefficients already present in r . The uniform approximation follows from positivity, integral one, and concentration at the origin of the kernels

$$F_m(u) = \frac{1}{m} \left| \sum_{j=0}^{m-1} e^{iju} \right|^2;$$

outside any fixed neighborhood of zero, $F_m(u) \leq 1 / [m \sin^2(u/2)]$. If θ annihilates every character in $L(H)$, each of these approximating polynomials takes the same value at θ and zero. Passing to the uniform limit yields $r(\theta) = r(0) = 0$, proving (B.1).

Choose finite generators $k_1, \dots, k_r$ of $L(H)$. For each fixed generator, Hausdorff convergence implies that all values of $e^{ik_i \cdot \theta}$, with $\theta \in H_j$, eventually lie in a proper small arc around one. But its image on H_j is either the singleton $\{1\}$ or the entire circle: it is parametrized by $e^{i(k_i \cdot a^{(j)})x}$. The latter possibility is excluded. Therefore every generator annihilates H_j for all sufficiently large j , simultaneously. By (B.1), $H_j \subset H$ eventually.

Finally set $\Phi(\theta) = -\sum_{i=1}^N \cos \theta_i$. Uniform continuity and Hausdorff convergence give

$$\max_H \Phi = \lim_j \max_{H_j} \Phi = \kappa_N.$$

For every sufficiently large j , inclusion gives $K(A_j) = \max_{H_j} \Phi \leq \kappa_N$. The definition of the infimum gives the reverse inequality. Hence these original admissible sets attain κ_N . $\square$

The limit subgroup is an auxiliary compactness device; it need not itself be a one-dimensional orbit with distinct positive coordinate frequencies. The proof avoids that difficulty by recovering equality on original members of the sequence. It provides neither a search cutoff nor a stopping rule for a frequency enumeration.

C. Exact moments and the polynomial hierarchy

Write $M_r(A) = E f_A^r$, including $M_0 = 1$. With $S = A \cup (-A)$, expansion and Fourier orthogonality give

$$M_r = 2^{-r} \# \{ (s_1, \dots, s_r) \in S^r : s_1 + \dots + s_r = 0 \}, r \geq 1.$$

Equation (C.1)

The tuples are ordered and repetitions are allowed. Thus every moment is rational, and its numerator is an exact count of integer relations. Equality modulo a finite integer is not the condition in (C.1).

For $d \in \mathbb{Z}_{\geq 0}$, with p ranging over real polynomials, define

$$B_d(A) = - \min_{p \neq 0, \deg p \leq d} \frac{E [f_A p(f_A)]^2}{E [p(f_A)]^2}.$$

Equation (C.2)

Only a polynomial of the *value of the original sum* is introduced. This construction does not change any amplitude or phase in f_A .

Proposition C.1 — A finite, attained optimization problem. Let

$$H_d = (M_{i+j})_{i,j=0}^d, F_d = (M_{i+j+1})_{i,j=0}^d.$$

The matrix H_d is positive definite, and B_d is the negative of the smallest generalized eigenvalue of (F_d, H_d) . In particular,

$$0 = B_0 \leq B_1 \leq B_2 \leq \dots \leq K(A).$$

Equation (C.3)

Proof. If u is the coefficient vector of p , the denominator and numerator in (C.2) are $u^T H_d u$ and $u^T F_d u$. The continuous nonconstant function f_A takes every value in the nondegenerate interval $[-K(A), N]$. A nonzero polynomial cannot vanish on this interval. At some point $p(f_A) \neq 0$, and by continuity its square is positive on a set of positive measure. Thus $u^T H_d u > 0$ for nonzero u .

The normalization $u^T H_d u = 1$ is a compact ellipsoid, so the minimum is attained. Standard diagonalization of $H_d^{-1/2} F_d H_d^{-1/2}$ gives the eigenvalue formulation. Constants give quotient zero, enlarging the degree enlarges the admissible class, and every quotient is at least $-K(A)$. These observations prove (C.3). $\square$

Let ν_A be the pushforward of μ under $x \mapsto f_A(x)$. Equation (C.2) is the univariate measure-based hierarchy applied to ν_A . Allowing a sum of squares $\sum_j p_j(t)^2$ instead of one square does not improve its optimum: its quotient is a weighted average of the individual quotients, with nonnegative weights $E p_j(f_A)^2$. This makes the connection to [Lasserre2021] exact, rather than merely analogous.

Proposition C.2 — Pointwise convergence in the frequency set. For each fixed admissible A , $B_d(A) \rightarrow K(A)$.

Proof. Set $m = \min f_A$, and choose $p_d(t) = (N - t)^d$. Fix $0 < \eta < N - m$. The set $E = \{f_A \leq m + \eta/2\}$ has positive measure. The unnormalized squared weight is at least $(N - m - \eta/2)^{2d}$ on E , and at most $(N - m - \eta)^{2d}$ on $\{f_A \geq m + \eta\}$. Consequently the normalized weight of the latter set tends to zero. Since $0 \leq f_A - m \leq 2N$, the corresponding Rayleigh quotient has limit superior at most $m + \eta$. Let $\eta \downarrow 0$, then use $B_d \leq K$. $\square$

This limit does not assert a rate uniform in A . Sections D and E provide a particular uniform rate. Neither statement supplies a lower bound of order $\sqrt{N}$ for the limiting value.

C.1. The first nontrivial level and a useful special class

The two matrices at $d = 1$ are

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & M_2 \end{pmatrix}, F_1 = \begin{pmatrix} 0 & M_2 \\ M_2 & M_3 \end{pmatrix}.$$

The least k for which $F_1 + k H_1$ is positive semidefinite is the positive root of $M_2 k^2 + M_3 k - M_2^2 = 0$. Hence

$$B_1(A) = \frac{\sqrt{M_3^2 + 4M_2^3} - M_3}{2M_2}.$$

Equation (C.4)

If $R(A) = \#\{(a, b) \in A^2 : a + b \in A\}$, then $M_3 = 3R(A)/4$: there are $6R(A)$ ordered signed zero-sum triples in (C.1). Thus, whenever $A \cap (A + A) = \emptyset$,

$$K(A) \geq B_1(A) = \sqrt{N/2}.$$

Equation (C.5)

This is a result for an explicitly restricted class. The condition forbids $a + a = b$ as well as relations involving three different frequencies.

D. A phase-mass lemma independent of the highest frequency

A bound obtained from $\|f_A'\|_\infty \leq \sum a$ deteriorates when the frequencies are large. The following different argument depends on the number of harmonics and not their magnitudes. It retains their common phase parameter throughout.

Lemma D.1 — Uniform mass near a minimum. Let $m = \min f_A$, let $\delta > 0$, and choose an integer $M \geq 2\pi N / \delta$. Then

$$\mu\{x : f_A(x) \leq m + \delta\} \geq M^{-N}.$$

Equation (D.1)

Proof. Partition $[0, 2\pi)^N$ into M^N half-open cubes of side $2\pi / M$. Pull this partition back to T by

$$x \mapsto (a_1 x, \dots, a_N x) \pmod{2\pi}.$$

One preimage E has measure at least M^{-N} . Choose $x_0 \in E$, and a minimizer x_* of f_A . For each $x \in E$, every coordinate of $a(x - x_0)$ modulo 2π has a representative of absolute value at most $2\pi / M$, because the two phase points lie in the same cube. Since cosine is 1-Lipschitz,

$$f_A(x + x_* - x_0) \leq f_A(x_*) + \frac{2\pi N}{M} \leq m + \delta.$$

Translation preserves μ . The translate of E lies in the asserted sublevel set and retains its measure. $\square$

A phase cube is used only to locate a set of common times with similar relative phases. No step replaces the one-dimensional orbit by a product distribution of independent phases. The mass bound can be extremely small; it is a guarantee, not a claim that the minimum is easy to encounter in a short experiment.

E. Uniform approximation using a Chebyshev filter

Theorem E.1 — A sufficient degree depending only on N and accuracy. Suppose $N \geq 1$ and $0 < \varepsilon \leq N$. Define

$$M = \lceil \frac{8\pi N}{\varepsilon} \rceil, d_* = \lceil \sqrt{\frac{N}{\varepsilon}} \left[N \log M + \log \left(\frac{16N}{\varepsilon} \right) \right] \rceil.$$

Equation (E.1)

For every integer $d \geq d_*$ and every admissible A with $|A| = N$,

$$0 \leq K(A) - B_d(A) \leq \varepsilon.$$

Equation (E.2)

Here and below $\log$ is natural logarithm. There is no restriction on $\max A$.

Proof. Set $m = \min f_A$ and

$$D = N - m - \varepsilon/2, u(t) = \frac{2t - (N + m + \varepsilon/2)}{D}.$$

Because $-N \leq m < 0$ and $\varepsilon \leq N$, we have $0 < N/2 < D \leq 2N$. The affine map v sends $[m + \varepsilon/2, N]$ to $[-1, 1]$. Choose

$$p(t) = T_d(v(t)),$$

where T_d is the Chebyshev polynomial of the first kind. The identities

$$T_d(\cos \theta) = \cos(d\theta), T_d(-\cosh v) = (-1)^d \cosh(dv)$$

Equation (E.3)

follow, for example, from $T_d((z + z^{-1})/2) = (z^d + z^{-d})/2$ [NISTDLMF]. They imply $|p(f_A)| \leq 1$ wherever $f_A \geq m + \varepsilon/2$.

On the smaller sublevel set $G = \{f_A \leq m + \varepsilon/4\}$,

$$v(f_A) \leq -1 - \frac{\varepsilon}{2D} \leq -1 - \frac{\varepsilon}{4N}.$$

Put $s = \varepsilon/(4N)$, so $0 < s \leq 1/4$. Direct integration gives

$$\operatorname{arcosh}(1+s) = \int_0^s \frac{dv}{\sqrt{v(v+2)}} \geq \frac{2\sqrt{s}}{\sqrt{2+s}} \geq \sqrt{s}.$$

Equation (E.4)

The last inequality holds for $s \leq 2$, hence in the stated domain. By (E.3), on G ,

$$|p(f_A)| \geq \frac{1}{2} \exp\left(\frac{d}{2} \sqrt{\frac{\varepsilon}{N}}\right).$$

Lemma D.1 with $\delta = \varepsilon/4$ yields $\mu(G) \geq M^{-N}$. Thus the normalization satisfies

$$W := E p(f_A)^2 \geq \frac{1}{4} M^{-N} \exp\left(d \sqrt{\frac{\varepsilon}{N}}\right).$$

Equation (E.5)

On $\{f_A < m + \varepsilon/2\}$, the excess $f_A - m$ is at most $\varepsilon/2$. On its complement, $f_A - m \leq 2N$ and $p(f_A)^2 \leq 1$. Splitting this expectation gives

$$\frac{E[(f_A - m)p(f_A)^2]}{W} \leq \frac{\varepsilon}{2} + \frac{2N}{W}.$$

The remaining term satisfies

$$\frac{\varepsilon}{2} + \frac{2N}{W} \leq \frac{\varepsilon}{2} + 8NM^N \exp\left(-d\sqrt{\frac{\varepsilon}{N}}\right) \leq \varepsilon.$$

Equation (E.6)

The final inequality is precisely the degree condition (E.1). The quotient for this admissible p is at most $m + \varepsilon$, so the optimized value satisfies $B_d \geq K - \varepsilon$. Its opposite bound follows from Proposition C.1. $\square$

E.1. Why the proof is not a circular algorithm

The comparison polynomial used in the proof contains the unknown minimum m . It proves *existence* of a polynomial in the admissible space with the required quotient. Computing B_d means optimizing over that whole space through the matrices F_d, H_d , whose entries are the exact counts (C.1); its definition does not require m . The proof does not claim an explicit list of optimal coefficients without performing that optimization.

The matrix dimension is $d + 1$, but a small dimension alone does not imply a low total cost. Moment enumeration, large integer sizes, conditioning, and exact eigenvalue certification all remain relevant. No large matrix at the worst-case sufficient degree was evaluated merely by proving this theorem.

E.2. A completely rational sufficient certificate

For a rational $\varepsilon > 0$, no rounded logarithm, square root, or value of π is necessary to certify a sufficient degree. Choose

$$M_0, s_0 \in \mathbb{Z}_{>0}, q, r \in \mathbb{Z}_{\geq 0},$$

such that

$$M_0 \varepsilon \geq 32N, s_0^2 \varepsilon \geq N, 2^q \geq M_0, 2^r \varepsilon \geq 16N.$$

Equation (E.7)

Then

$$d_0 = s_0(Nq + r)$$

Equation (E.8)

is sufficient for (E.2). Indeed, $\pi < 4$ makes M_0 a permitted choice in the phase-mass estimate, and positivity of s_0 gives

$$d_0 \sqrt{\varepsilon/N} \geq Nq + r.$$

Furthermore,

$$e^{Nq+r} \geq 2^{Nq+r} \geq M_0^N \frac{16N}{\varepsilon}.$$

Substitution into (E.6) proves the claim. The requirement $s_0 > 0$ is essential; the squared inequality by itself would allow a negative integer and would not imply the needed inequality for the exponent.

E.3. Comparison with the elementary power filter

For $0 < \varepsilon \leq 1$, the earlier choice $p(t) = (N - t)^d$ gives the sufficient degree

$$d \geq \lceil \frac{2N^2}{\varepsilon} \log \lceil \frac{4\pi N}{\varepsilon} \rceil \rceil.$$

Equation (E.9)

Here is a short derivation. Let $g = N - f_A$, $G = \max g = N + K$, and $r_0 = 2d$. Lemma D.1 gives

$$\|g\|_{r_0} \geq (G - \varepsilon/2) \exp\left(-\frac{N \log M'}{r_0}\right), M' = \lceil \frac{4\pi N}{\varepsilon} \rceil.$$

Using $G \leq 2N$ and $1 - e^{-u} \leq u$ yields

$$G - \|g\|_{r_0} \leq \frac{\varepsilon}{2} + \frac{2N^2 \log M'}{r_0} \leq \varepsilon.$$

Hölder's inequality gives $E g^{r_0+1} / E g^{r_0} \geq \|g\|_{r_0}$. The quotient associated with $(N - t)^d$ therefore proves (E.9).

The sufficient order improves from $O(N^2 \varepsilon^{-1} \log(2N/\varepsilon))$ to $O(N^{3/2} \varepsilon^{-1/2} \log(2N/\varepsilon))$. This is a comparison of the two bounds proved in this report, not a claim to improve the best known general polynomial-optimization theory. In particular, [SlotLaurent2021] studies convergence of the same pushforward approach using polynomial concentration. Its constants and geometric hypotheses must be compared carefully before any claim of novelty for (E.1).

For $\varepsilon = \alpha \sqrt{N}$ with fixed $0 < \alpha \leq 1$, (E.1) gives $d = O_\alpha(N^{5/4} \log(2N))$. Approximating an unknown minimum to that accuracy still does not prove that its magnitude is at least a positive multiple of $\sqrt{N}$.

F. Consecutive frequencies and low-degree certificates

The interval family $A_N = \{1, \dots, N\}$ is especially useful for testing a proposed certificate. It has strong arithmetic relations and a directly visible deep negative excursion.

Put $L = 2N + 1$. Summing a geometric progression gives

$$D_N(x) := 1 + 2f_{A_N}(x) = \sum_{a=-N}^N e^{iax} = \frac{\sin(Lx/2)}{\sin(x/2)},$$

Equation (F.1)

with its continuous value L at multiples of 2π . At $x = 3\pi/L$,

$$K(A_N) \geq \frac{1}{2} + \frac{1}{2 \sin(3\pi/(2L))} \geq \frac{1}{2} + \frac{L}{3\pi}.$$

Equation (F.2)

The final step uses $\sin u \leq u$ with $0 < u \leq \pi/2$. Thus any certificate that stays bounded on this family is missing an actual minimum of at least linear depth.

F.1. Exact formulas for all required moments

For an integer $r \geq 1$, inclusion–exclusion gives

$$C_r(N) := E D_N^r = \sum_{j=0}^{\lfloor r/2 \rfloor} (-1)^j \binom{r}{j} \binom{(r-2j)N+r-1-j}{r-1}, C_0(N) = 1.$$

Equation (F.3)

Set $\binom{u}{v} = 0$ for integers $0 \leq u < v$. To prove (F.3), shift each exponent in a zero-sum tuple from $[-N, N]$ to $[0, 2N]$. The sum of the shifted entries must be rN . Unrestricted nonnegative solutions are counted by stars and bars; choosing j entries that exceed $2N$ and subtracting $2N+1$ from them gives the displayed sum. Terms beyond $\lfloor r/2 \rfloor$ cannot contribute. Then

$$M_r(N) = 2^{-r} \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} C_j(N).$$

Equation (F.4)

In particular,

$$M_2 = \frac{N}{2}, M_3 = \frac{3N(N-1)}{8}, M_4 = \frac{N(8N^2-6N+7)}{24}.$$

Equation (F.5)

For $r \geq 2$, the leading coefficient is

$$\lim_{N \rightarrow \infty} \left(\frac{M_r(N)}{N^{r-1}} \right) = \frac{1}{2^r (r-1)!} \sum_{j=0}^{\lfloor r/2 \rfloor} (-1)^j \binom{r}{j} (r-2j)^{r-1}.$$

Equation (F.6)

For $r = 2, 3, 4, 5, 6, 7$, these values are respectively

$$\frac{1}{2}, \frac{3}{8}, \frac{1}{3}, \frac{115}{384}, \frac{11}{40}, \frac{5887}{23040}.$$

These are finite combinatorial identities, not a conclusion from simulated limiting distributions.

F.2. The first two levels can miss the scale

Equation (C.4) and (F.5) give $B_1(A_N) \rightarrow 2/3$. The second level also stays bounded:

$$B_2(A_N) \rightarrow \frac{24}{11}.$$

Equation (F.7)

To check the latter assertion, use the basis $(1, f/N, (f/N)^2)$. For fixed k , the lower-right block of $F_2 + kH_2$ tends to

$$Q = \begin{pmatrix} 3/8 & 1/3 \\ 1/3 & 115/384 \end{pmatrix}, \det Q = \frac{11}{9216} > 0,$$

and the off-diagonal vector linking it to the first coordinate tends to $h = (1/2, 3/8)^T$. The first diagonal entry is k ; all remaining contributions from kH_2 tend to zero. The limiting Schur complement is

$$k - h^T Q^{-1} h = k - \frac{24}{11}.$$

For every fixed $k > 24/11$, the matrix is positive definite for all sufficiently large N ; for every fixed $k < 24/11$, it is not positive semidefinite for all sufficiently large N . The threshold characterization of B_2 proves (F.7).

F.3. A cubic polynomial detects a linear depth on this family

Choose

$$p_N(t) = t(1 - t/N)(1 - 2t/N).$$

Direct substitution of (F.3)–(F.4) in its Rayleigh quotient gives

$$B_3(A_N) \geq R_N = \frac{56N^5 + 1005N^4 + 5750N^3 + 12495N^2 + 11474N + 3780}{12(43N^4 + 465N^3 + 1175N^2 + 1125N + 372)}.$$

Equation (F.8)

To make this algebra independently reproducible, write

$$\begin{aligned} U_N &= M_3 - 6M_4/N + 13M_5/N^2 - 12M_6/N^3 + 4M_7/N^4, \\ V_N &= M_2 - 6M_3/N + 13M_4/N^2 - 12M_5/N^3 + 4M_6/N^4. \end{aligned}$$

The identities being used are exactly

$$\begin{aligned} -N^4 U_N &= \frac{56N^6 + 1005N^5 + 5750N^4 + 12495N^3 + 11474N^2 + 3780N}{5760}, \\ N^4 V_N &= \frac{43N^5 + 465N^4 + 1175N^3 + 1125N^2 + 372N}{480}. \end{aligned}$$

They follow by polynomial expansion of the finite binomial formulas above. Every coefficient of V_N is positive. Subtracting $(14/129)N(N^4 V_N)$ from $-N^4 U_N$ leaves, in ascending powers of N , the coefficients

$$0, \frac{21}{32}, \frac{47255}{24768}, \frac{31619}{16512}, \frac{18145}{24768}, \frac{1145}{16512}, 0.$$

Consequently $R_N > (14/129)N$ for every $N \geq 1$. The direct bound (F.2) has a larger linear coefficient. The cubic calculation is useful because it demonstrates a change of scale within the hierarchy, not because it improves the interval's elementary bound. No analogous all-set lower bound for B_3 is established here.

G. The lattice operator and a finite open box

Let T be the bilateral shift on $\ell^2(\mathbb{Z})$,

$$(T\psi)(n) = \psi(n-1), H_A = \frac{1}{2} \sum_{a \in A} (T^a + T^{-a}).$$

Equation (G.1)

This is a bounded self-adjoint operator with norm at most N . In the language of a tight-binding model, the admissible hopping distances are the elements of A . The factor $1/2$ ensures that the two opposing directions contribute one unit-amplitude cosine after Fourier transformation. This is a mathematical model; arbitrary long hopping distances are not asserted to occur in a particular material.

Proposition G.1 — Exact spectral identity.

$$\text{spec}(H_A) = [-K(A), N], -K(A) = \inf_{\|\psi\|_2=1} \langle \psi, H_A \psi \rangle.$$

Equation (G.2)

Proof. The unitary Fourier map $U\psi(x) = \sum_n \psi(n)e^{inx}$ from $\ell^2(\mathbb{Z})$ to $L^2(T, \mu)$ sends T to multiplication by e^{ix} . It sends H_A to multiplication by $f_A(x)$. The spectrum of a multiplication operator is its essential range. For this continuous function, every neighborhood of an attained value has a preimage of positive measure, so essential range and range agree. The range is the whole interval between its minimum and maximum by continuity and connectedness of T . The Rayleigh infimum is at least the minimum; normalized indicators of small neighborhoods of a minimizing point approach it from above. $\square$

For the localized state δ_0 ,

$$\langle \delta_0, H_A \delta_0 \rangle = 0, \langle \delta_0, H_A^2 \delta_0 \rangle = N/2.$$

Moreover, $U p(H_A) \delta_0 = p(f_A)$. The quotient in (C.2) is exactly the energy expectation of this filtered state after normalization. This gives a useful physical interpretation of the polynomial filter without changing the original waveform.

Discreteness of the lattice does not imply a discrete spectrum here. A nonconstant trigonometric polynomial takes any particular value on only finitely many points of the circle. A nonzero eigenfunction of multiplication by f_A would have to be supported on such a set of measure zero, which is impossible in L^2 . The spectral edge is approached by states; it is not a normalizable isolated ground state of this infinite translation-invariant operator.

G.1. Open boundaries and the exact loss at an edge

Restrict the lattice to $0, \dots, L-1$, for an integer $L \geq 1$, and let $H_{A,L}$ have entry $1/2$ between indices i, j when $|i-j| \in A$, and zero otherwise. Put

$$B_L^{box}(A) = -\lambda_{\min}(H_{A,L}).$$

Proposition G.2 — Finite-box bound.

$$0 \leq B_L^{box} \leq B_{L+1}^{box} \leq K(A), 0 \leq K(A) - B_L^{box}(A) \leq \sum_{a \in A} \min(a/L, 1).$$

Equation (G.3)

Proof. The matrix has trace zero, so its smallest eigenvalue is nonpositive. Extending a vector by zero embeds each box into the next and into the full lattice; the variational principle proves the first chain. For the normalized vector $\psi_x(n) = e^{inx} / \sqrt{L}$ in the box, counting the $(L-a)_+$ surviving pairs at distance a gives

$$\langle \psi_x, H_{A,L} \psi_x \rangle = \sum_{a \in A} (1 - a/L)_+ \cos(ax).$$

Its difference from $f_A(x)$ has absolute value at most the sum in (G.3). Evaluate at a minimizer of f_A and use the minimum-eigenvalue variational principle again. $\square$

For a fixed set A , the right side tends to zero. If $L > \max A$, it equals $\sum a/L$. However, no fixed box length depending only on N provides a positive certificate for all unrestricted frequencies: for $A = \{a\}$ and $a \geq L$, there are no surviving edges, so $B_L^{box} = 0$ although $K(A) = 1$.

For comparison, if all frequencies divided by their gcd are odd, evaluation at the corresponding half-period makes every cosine equal -1 , and $K = N$. In the lattice model the parity transformation $\psi(n) \mapsto (-1)^n \psi(n)$ changes the sign of every such hopping term. This elementary favorable case cannot be assumed for arbitrary sets.

H. A discrete grid can miss the minimum

Theorem H.1 — Exact energy does not imply an exact negative peak. For $A_N = \{1, \dots, N\}$, let $q = 2N + 1$ and $x_j = 2\pi j/q$, $0 \leq j < q$. Then

$$f_{A_N}(x_0) = N, f_{A_N}(x_j) = -\frac{1}{2} (1 \leq j < q),$$

Equation (H.1)

while

$$\frac{1}{q} \sum_{j=0}^{q-1} f_{A_N}(x_j)^2 = \frac{N}{2}, K(A_N) \geq \frac{1}{2} + \frac{q}{3\pi}.$$

Equation (H.2)

Proof. The terms in $D_N(x_j)$ from (F.1) run through a complete set of q -th roots of unity. Their sum is zero for $j \neq 0$, and q for $j = 0$. This proves (H.1). The square average equals $(N^2 + 2N/4)/(2N+1) = N/2$. The final bound is (F.2). $\square$

Thus the grid depth is $1/2$ while the continuous depth grows at least linearly. This does not mean the grid has lost the information needed for exact interpolation. A trigonometric polynomial of degree N is uniquely determined by these $2N+1$ values: their discrete Fourier transform recovers its coefficients. The mistake is to identify the minimum of the interpolated curve with the minimum of the samples. Interpolation is not a convex averaging rule, and values between samples can fall below every sample.

The finite cyclic model makes this discrepancy especially transparent. Let T_0 be the cyclic shift on q coordinates. Since the residues $\pm 1, \dots, \pm N$ are exactly the nonzero residues modulo q ,

$$H_0 = \frac{1}{2} \sum_{a=1}^N (T_0^a + T_0^{-a}) = \frac{J - I}{2},$$

where J has every entry equal to one. Its eigenvalues are N , once, and $-1/2$, with multiplicity $2N$. For every integer $k \geq 0$,

$$2^k \frac{1}{q} \sum_j f_{A_N}(x_j)^k = \frac{(2N)^k + 2N(-1)^k}{2N+1} \in \mathbb{Z}.$$

Equation (H.3)

The divisibility follows from $2N \equiv -1 \pmod{2N+1}$. These quantities also count closed walks from a vertex in the complete graph. Therefore positivity of moment matrices, the correct second moment, and integrality of the doubled-signal moments can coexist with a grid depth of only $1/2$. Those conditions alone do not enforce the exact integer relations in (C.1).

H.1. Twisting the boundary restores the continuous spectrum

For a real ϕ , define a unitary shift on the same basis by

$$T_\phi e_j = e_{j+1} (j < q-1), T_\phi e_{q-1} = e^{i\phi} e_0.$$

Then $T_\phi^q = e^{i\phi} I$. Its eigenvalues are the q distinct roots $z_j = \exp(i(2\pi j + \phi)/q)$. For any admissible A , retain the actual integer powers and define

$$H_\phi = \frac{1}{2} \sum_{a \in A} (T_\phi^a + T_\phi^{-a}).$$

Its eigenvalues are exactly

$$E_j(\phi) = f_A\left(\frac{2\pi j + \phi}{q}\right).$$

As ϕ ranges over $[0, 2\pi]$, these arguments cover the whole circle. Hence

$$\min_{0 \leq \phi \leq 2\pi} \lambda_{\min}(H_\phi) = \min_x f_A(x) = -K(A).$$

Equation (H.4)

For the interval example, $\phi = \pi$ and $j = 1$ already give the deep witness $x = 3\pi/q$. For frequencies exceeding q , reducing exponents modulo q without keeping the accumulated factor $e^{i\phi}$ would be an error. The formula requires the actual powers.

Equation (H.4) is an exact reformulation, not a proof that some boundary twist always yields energy below $-c\sqrt{N}$. That last assertion is still the original unresolved lower-bound problem. The physical use of tunable twisted boundary conditions has established precedents; for example, Bardyn, Huber, and Zilberberg, *New Journal of Physics* 16, 123013 (2014), arXiv:1312.6894. Our formulas follow directly from the finite shift defined above.

H.2. Certified errors for a general sampling grid

For an integer $M \geq 1$, sample the original function at $2\pi j/M$, and let $K_M = -\min_{0 \leq j < M} f_A(2\pi j/M)$. Define $S_1 = \sum_{a \in A} a$ and $S_2 = \sum_{a \in A} a^2$. Then

$$K_M \leq K(A) \leq K_M + \min\left(\frac{\pi S_1}{M}, \frac{\pi^2 S_2}{2M^2}\right).$$

Equation (H.5)

To prove this, take a true minimizer x_* and its nearest grid point, at circular distance at most π/M . The derivative satisfies $|f_A'| \leq S_1$, giving the first error bound by the mean value theorem. At a minimizing point, $f_A'(x_*) = 0$; Taylor's theorem and $|f_A''| \leq S_2$ give the second. The grid minimum is no greater than the value at this particular grid point, and no smaller than the true minimum. These two comparisons prove (H.5). The lower endpoint may be replaced by $\max(1, K_M)$ using Proposition A.1. Unlike Theorem E.1, these sampling bounds explicitly depend on the sizes of the frequencies.

I. Why integer amplitudes do not preserve the conjecture

A tempting extension replaces equal amplitudes by positive weights and asks for an absolute $c > 0$ such that

$$-\min_x \sum_{a=1}^N w_a \cos(ax) \geq c \left(\sum_{a=1}^N w_a^2 \right)^{1/2}.$$

Equation (I.1)

The normalization reflects the RMS, whose square is $\sum w_a^2/2$. This extension is false even for positive integer weights.

Proposition I.1 — The Fejér amplitude barrier. Put $w_a = 1 - a/(N+1)$, $1 \leq a \leq N$. Then

$$-\min_x \sum_{a=1}^N w_a \cos(ax) = \frac{1}{2}, \sum_{a=1}^N w_a^2 = \frac{N(2N+1)}{6(N+1)}.$$

Equation (I.2)

Proof. Expanding a finite squared modulus gives

$$\frac{1}{N+1} \left| \sum_{j=0}^N e^{ijx} \right|^2 = 1 + 2 \sum_{a=1}^N \left(1 - \frac{a}{N+1} \right) \cos(ax) \geq 0.$$

At each nontrivial $(N+1)$ -st root of unity, the geometric sum vanishes, so the bound $-1/2$ is attained. For the norm, set $j = N+1-a$ and use $\sum_{j=1}^N j^2 = N(N+1)(2N+1)/6$. $\square$

Multiplication by $N+1$ produces integer amplitudes $v_a = N+1-a$. Their negative peak equals $(N+1)/2$, and

$$\frac{(N+1)/2}{\sqrt{\sum v_a^2}} = \sqrt{\frac{3(N+1)}{2N(2N+1)}} \rightarrow 0.$$

Equation (I.3)

This disproves (I.1) for every proposed positive absolute constant. A fixed amplitude quantum merely multiplies both numerator and denominator and cannot remove the obstruction.

These weights cannot be relabeled as distinct unit-amplitude frequencies: repeated sinusoids at the same frequency combine into a larger coefficient. The original assumption is equal nonzero coefficients at distinct frequencies, which is much stronger than requiring a discrete amplitude step. Likewise, a quantum occupation number does not by itself specify the signed mean field of a harmonic component; the state and its phase information matter. No theorem for (A.1) automatically extends to arbitrary music with unequal amplitudes and phases.

J. A positive Riesz-product certificate

Define

$$P_A(x) = \prod_{a \in A} (1 - \cos(ax)), Z_A = E P_A, B_R(A) = -\frac{E[f_A P_A]}{Z_A}.$$

Equation (J.1)

The product is nonnegative and positive outside a finite union of zeros, so $Z_A > 0$. It follows immediately that $B_R(A) \leq K(A)$. This proof does not establish that $B_R \geq 0$ for every set; $\max(0, B_R)$ remains a valid certificate even if a negative value occurs. The use of positive Riesz products is classical, and no new general inequality for them is claimed here.

Put

$$Q_A(z) = \prod_{a \in A} (2 - z^a - z^{-a}) = \sum_j q_j z^j.$$

Its coefficients are integers with $q_j = q_{-j}$, and $Q_A(e^{ix}) = 2^N P_A(x)$. Extracting constant coefficients proves

$$Z_A = \frac{q_0}{2^N}, B_R(A) = -\frac{\sum_{a \in A} q_a}{q_0}.$$

Equation (J.2)

An independent implementation can use the identity $Q_A(e^{ix}) = \left| \prod_{a \in A} (1 - e^{iax}) \right|^2$ and integer autocorrelation. No sampling of trigonometric values is needed for either formula.

For $A = \{1, 4, \dots, 4^{N-1}\}$, signed digit representations with digits $-1, 0, 1$ are unique. Indeed, subtracting two representations leaves digits between -2 and 2 ; the largest nonzero term has magnitude at least 4^m , while the total possible magnitude of all smaller terms is $2(4^m - 1)/3 < 4^m$. Thus the only constant contribution in P_A is one and the coefficient at each $e^{\pm i 4^j x}$ is $-1/2$. Consequently

$$Z_A = 1, B_R(A) = N/2.$$

Equation (J.3)

Short additive relations change this calculation. For $A = \{1, 2, 3\}$, exact multiplication gives $q_0 = 6$ and $(q_1, q_2, q_3) = (0, -3, -2)$. Hence $B_R = 5/6 < \sqrt{3}/2$, since $25/36 < 3/4$. The conjectured coefficient $1/2$ for this detector is therefore false.

A stronger finite example is

$$A = \{1, 2, 3, 4, 5, 7, 9, 11, 13\}.$$

Here $q_0 = 20$, and the coefficients at the listed frequencies are

$$(4, -2, -7, -10, -2, 4, -1, -3, -4).$$

Their sum is -21 , so $B_R = 21/20$ and $B_R/\sqrt{9} = 7/20$. This excludes any larger universal coefficient for this particular detector.

Finite computational result. The archived exact enumeration of all $2^{18} - 1 = 262,143$ nonempty subsets of $\{1, \dots, 18\}$ finds the minimum ratio $B_R/\sqrt{|A|}$ to be exactly $7/20$. Comparisons use the exact sign and the rational number $B_R^2/|A|$, not rounded square roots. A second coefficient method was compared for all 4,095 nonempty subsets of $\{1, \dots, 12\}$. This is a finite-domain minimum. It neither proves $7/20$ valid for unrestricted sets nor rules out a smaller positive universal coefficient. It is also a result about B_R , not the actual extremum K .

K. Negative moments, quantiles, and a saturating detector

Let $Y_A = (f_A)_- = \max(-f_A, 0)$. Every probability density $\rho \geq 0$ with $E\rho = 1$ yields a valid bound $-E(\rho f_A) \leq K(A)$. Many ideas in the investigation amount to selecting such a density, or to measuring a norm or quantile of Y_A . Their validity as lower bounds must be distinguished from their ability to retain a uniform square-root scale.

K.1. The first absolute moment and the fourth moment

Since $E f_A = 0$, its positive and negative parts have equal integrals. Therefore

$$K(A) \geq E Y_A = \frac{1}{2} \|f_A\|_1.$$

Equation (K.1)

Hölder's inequality, applied to $|f|^{2/3}|f|^{4/3}$, gives

$$M_2 \leq \|f\|_1^{2/3} M_4^{1/3}, K(A) \geq \frac{M_2^{3/2}}{2\sqrt{M_4}} = \frac{N^{3/2}}{4\sqrt{2}\sqrt{M_4}}.$$

Equation (K.2)

A bound $M_4 / M_2^2 \leq C$ by an absolute constant would make (K.2) a square-root certificate on that class. It is false as a uniform assumption: (F.5) gives $M_4 / M_2^2 \sim 4N/3$ for intervals.

There is an arithmetic caution in interpreting the fourth moment. Define the ordered counts

$$E(A) = \#\{(a, b, c, d) \in A^4 : a + b = c + d\},$$

$$T(A) = \#\{(a, b, c, d) \in A^4 : a + b + c = d\}.$$

Separating two-positive/two-negative and three-positive/one-negative signed tuples in (C.1) yields

$$M_4 = \frac{3}{8} E(A) + \frac{1}{2} T(A).$$

Equation (K.3)

The second term cannot simply be omitted. There are six sign placements in the first case and eight in the second, followed by division by 2^4 .

On the interval family, the elementary estimate

$$|D_N(x)| \leq \min(L, \pi/|x|), 0 < |x| \leq \pi,$$

Equation (K.4)

follows from $\sin(|x|/2) \geq |x|/\pi$. Splitting its integral at π/L gives

$$\|D_N\|_1 \leq 1 + \log L, EY_{A_N} \leq \frac{1}{2} + \frac{1}{4} \log L.$$

Equation (K.5)

Hence the certificate (K.1), divided by $\sqrt{N}$, tends to zero on a family whose true minimum satisfies (F.2). The obstruction concerns the detector, not the conjecture.

K.2. Higher negative moments

For $0 < p < \infty$, set $L_p^-(A) = (EY_A^p)^{1/p}$. For $p < 1$ this is a quasi-norm. On a probability space, Jensen's inequality gives

$$L_p^-(A) \leq L_q^-(A) \leq K(A) \quad (0 < p < q), \lim_{p \rightarrow \infty} L_p^-(A) = K(A).$$

Equation (K.6)

To prove the limit, every set $\{Y_A > K - \eta\}$ has positive measure; its contribution gives $L_p^-(A) \geq (K - \eta) \mu(\{Y_A > K - \eta\})^{1/p}$. Let $p \rightarrow \infty$, then $\eta \downarrow 0$.

For fixed p , the interval family has the orders

$$L_p^-(A_N) = \begin{cases} \Theta_p(1), & 0 < p < 1, \\ \Theta(\log N), & p = 1, \\ \Theta_p(N^{1-1/p}), & p > 1. \end{cases}$$

Equation (K.7)

Here constants may depend on the fixed p , and the orders refer to $N \rightarrow \infty$. We give the details because a change of order at $p = 1$ matters.

For $p > 1$, (K.4) implies

$$E|D_N|^p \leq L^{p-1} + \frac{L^{p-1} - 1}{p-1}.$$

Since $|f_{A_N}|^p \leq (|D_N|^p + 1)/2$, this proves the upper bound. For a lower bound, if $N \geq 2$, use the disjoint arcs

$$E_N = \left[\frac{8\pi}{3L}, \frac{10\pi}{3L} \right] \cup \left[-\frac{10\pi}{3L}, -\frac{8\pi}{3L} \right].$$

Equation (K.8)

On the positive arc, $Lx/2 \in [4\pi/3, 5\pi/3]$, so $\sin(Lx/2) \leq -1/2$, whereas $\sin(x/2) \leq 5\pi/(3L)$. The negative arc follows by evenness. Thus

$$\mu(E_N) = \frac{2}{3L}, Y_{A_N} \geq \frac{3L}{20\pi} \text{ on } E_N.$$

Equation (K.9)

Its contribution yields the matching lower order for every $p > 1$.

For $p = 1$, the upper bound is (K.5). To obtain the logarithmic lower bound, for $N \geq 6$ put $m_0 = \lfloor N/3 \rfloor$ and take the arcs

$$J_k = \left[\frac{(12k+7)\pi}{3L}, \frac{(12k+11)\pi}{3L} \right], 0 \leq k < m_0,$$

together with their reflections. They lie within $[-\pi, \pi]$. On each positive arc, $\sin(Lx/2) \leq -1/2$ and $Y_{A_N} \geq 1/(2x)$. Integration and $\log(b/a) \geq (b-a)/b$ give

$$EY_{A_N} \geq \frac{2}{\pi} \sum_{k=0}^{m_0-1} \frac{1}{12k+11} \geq \frac{1}{6\pi} \log \left(1 + \frac{12m_0}{11} \right) \geq \frac{1}{6\pi} \log \left(1 + \frac{2N}{11} \right).$$

For $0 < p < 1$, subadditivity of u^p and (K.4) give

$$EY_{A_N}^p \leq 2^{-p} \left(1 + \frac{1}{1-p} \right).$$

A uniform positive lower bound follows from a fixed mass of moderately negative values. On any interval of length ℓ , the set where $\sin y \leq 0$ has length at least $\ell/2 - \pi/2$: a primitive of $1_{\{\sin y \leq 0\}} - 1/2$ has oscillation $\pi/2$. On $x \in [\pi/2, \pi]$, substituting $y = Lx/2$ yields a suitable subset of length at least $\pi/4 - \pi/L \geq \pi/20$ for $L \geq 5$. Reflecting it gives

$$\mu\{f_{A_N} \leq -1/2\} \geq 1/20.$$

Equation (K.10)

For $N=1$ the measure is $1/3$, so (K.10) remains valid. Consequently $L_p^-(A_N) \geq (1/2)(1/20)^{1/p}$, completing (K.7).

It follows that every fixed $p < 2$ gives $L_p^-(A_N)/\sqrt{N} \rightarrow 0$. For $p=2$, the interval family has the correct scale; it neither establishes nor refutes a uniform negative-second-moment lower bound for all frequency sets.

K.3. A fixed probability of seeing the trough is too demanding

For $0 < p < 1$, define the upper-tail depth quantile

$$q_A(p) = \sup\{t \geq 0 : \mu(Y_A \geq t) \geq p\}.$$

It is nonincreasing in p , satisfies $0 \leq q_A(p) \leq K(A)$, and tends to $K(A)$ as $p \downarrow 0$. The last claim follows from the positive measure of every strict near-maximum neighborhood of Y_A .

For intervals and $t > 1/2$, (F.1) implies that $f_{A_N}(x) \leq -t$ requires $|x| \leq \pi/(2t-1)$, because $|f_{A_N}(x)| \leq 1/2 + \pi/(2|x|)$. Therefore

$$\mu(f_{A_N} \leq -t) \leq \min\left(1, \frac{1}{2t-1}\right), q_{A_N}(p) \leq \frac{1+1/p}{2}.$$

Equation (K.11)

Every fixed $p > 0$ gives a bounded quantile as N grows. In contrast, (K.9) proves

$$q_{A_N}\left(\frac{2}{3L}\right) \geq \frac{3L}{20\pi}.$$

A fraction of time of order $1/N$ can carry a trough of order N . Choosing a fixed fraction of observation time is therefore a substantially stronger requirement than the existence of a deep point.

K.4. The zero-threshold Fermi-Dirac detector

For $\beta \geq 0$, introduce the logistic weight

$$w_\beta(t) = \frac{1}{1+e^{\beta t}}, B_\beta^{FD}(A) = -\frac{E[f_A w_\beta(f_A)]}{E w_\beta(f_A)}.$$

Equation (K.12)

Only the mathematical shape of the Fermi-Dirac occupation function at zero chemical potential is borrowed. The waveform is not declared a fermionic energy, and β is a parameter rather than a measured temperature.

For all admissible A , $0 \leq B_\beta^{FD} \leq K(A)$. The upper bound follows from a normalized positive weight. For the lower bound, use $w_\beta(t) = (1 - \tanh(\beta t/2))/2$ and $E f_A = 0$; the negative numerator becomes

$\frac{1}{2} E[f_A \tanh(\beta f_A/2)] \geq 0$. At $\beta=0$ the value is zero. Since the zero set of f_A has measure zero, bounded convergence gives

$$\lim_{\beta \rightarrow \infty} B_{\beta}^{FD}(A) = \frac{E Y_A}{\mu(f_A < 0)}.$$

Equation (K.13)

This is the average depth of the negative part, not its largest depth.

Proposition K.1 — Failure uniformly over the tuning parameter. For every $N \geq 1$,

$$0 \leq \sup_{\beta \geq 0} B_{\beta}^{FD}(A_N) \leq 20 + 10 \log(2N + 1).$$

Equation (K.14)

Proof. On the set in (K.10), $w_{\beta}(f_{A_N}) \geq 1/2$, so its expectation is at least $1/40$, uniformly in β . Since $0 \leq w_{\beta} \leq 1$, the negative numerator is at most $E Y_{A_N}$. Apply (K.5) and multiply by 40. $\square$

Thus no selection of β , even one depending on the entire set, turns this zero-threshold family into an all-set square-root certificate. The restriction to zero threshold is crucial. A changing chemical-potential parameter defines a different family and is not excluded by this proof.

L. Gibbs weights and an entropy budget

Saturating weights need not adequately emphasize a rare deep trough. Exponential weights behave differently. Set $X = -f_A$, so $E X = 0$, $E X^2 = N/2$, and $\max X = K(A)$. Define

$$g(\beta) = \log E e^{\beta X}, \rho_{\beta} = e^{\beta X - g(\beta)}, \beta > 0.$$

Equation (L.1)

The normalized density ρ_{β} has expectation one. Differentiation under the integral is justified because X is bounded, and gives

$$g'(\beta) = E(\rho_{\beta} X), g''(\beta) = \text{Var}_{\rho_{\beta}}(X) > 0.$$

Equation (L.2)

Strict positivity holds because X is nonconstant and ρ_{β} is everywhere positive. For finite $\beta > 0$, Jensen's inequality and nonconstancy give

$$0 < \frac{g(\beta)}{\beta} < K(A).$$

This bound increases with β , since

$$\frac{d}{d\beta} \frac{g(\beta)}{\beta} = \frac{E \rho_{\beta} \log \rho_{\beta}}{\beta^2} > 0.$$

Moreover, it tends to $K(A)$. For each $\eta > 0$, the set $E_{\eta} = \{X \geq K - \eta\}$ has positive measure, and

$$\frac{g(\beta)}{\beta} \geq K - \eta + \frac{\log \mu(E_{\eta})}{\beta}.$$

Take $\beta \rightarrow \infty$, then $\eta \downarrow 0$. Again, a convergent description of the extremum is not a lower bound for its size.

L.1. Entropy-constrained optimization

For a density $\rho \geq 0$ with $E\rho = 1$, let

$$D(\rho) = E(\rho \log \rho), 0 \log 0 = 0.$$

For $h \geq 0$, put

$$b_A(h) = \sup \{ E(\rho X) : D(\rho) \leq h, \rho \geq 0, E\rho = 1 \}.$$

Equation (L.3)

This is an optimization over a redistribution of the original time measure. Entropy measures the cost of concentrating that measure; it is not extra physical energy in the waveform.

Theorem L.1 — Exact dual formula and attainment. For $h > 0$, there is a unique $\beta_h > 0$ such that

$$\beta_h g'(\beta_h) - g(\beta_h) = h.$$

Equation (L.4)

The optimizer in (L.3) is ρ_{β_h} , and

$$b_A(h) = g'(\beta_h) = \inf_{\beta > 0} \frac{h + g(\beta)}{\beta}.$$

Equation (L.5)

Also $b_A(0) = 0$, b_A is nondecreasing and concave, and $b_A(h) \rightarrow K(A)$ as $h \rightarrow \infty$.

Proof. Relative entropy is nonnegative by convexity of $u \log u$. Applied relative to the positive density ρ_β , this says

$$0 \leq D(\rho \parallel \rho_\beta) = D(\rho) - \beta E(\rho X) + g(\beta).$$

It proves the upper bound in (L.5) for every admissible ρ . The entropy of ρ_β is

$$D_\beta = \beta g'(\beta) - g(\beta), D_\beta' = \beta g''(\beta) > 0,$$

and tends to zero with $\beta \downarrow 0$.

We verify that $D_\beta \rightarrow \infty$. The nonconstant trigonometric polynomial X attains its maximum at finitely many points: otherwise the polynomial equation obtained after multiplication by a suitable power of e^{ix} would have infinitely many roots. Therefore $p_\eta = \mu(E_\eta) \rightarrow 0$ as $\eta \downarrow 0$. For a fixed $\eta > 0$, the Gibbs mass outside E_η is at most $p_{\eta/2}^{-1} e^{-\beta \eta/2}$, hence tends to zero. If the Gibbs mass of E_η is a , Jensen's inequality on this set and its complement gives

$$D_\beta \geq a \log(a/p_\eta) + (1-a) \log((1-a)/(1-p_\eta)) \geq a \log(1/p_\eta) - \log 2.$$

First let $\beta \rightarrow \infty$ for fixed η , then let $\eta \downarrow 0$. This proves divergence. Continuity and strict increase now imply a unique solution of (L.4). Substitution of ρ_{β_h} into the relative-entropy inequality makes it an equality, establishing attainment and (L.5).

At zero entropy only the uniform density is possible, so $b_A(0) = 0$. Increasing the budget enlarges the feasible set. A mixture of densities feasible at two budgets is feasible at the same mixture of budgets, by convexity of entropy; its objective is the corresponding mixture. This proves concavity. Finally, the density $1_{E_\eta}/p_\eta$ has entropy $\log(1/p_\eta)$ and objective at least $K - \eta$, proving the limit as the budget grows. $\square$

These variational identities belong to the established Gibbs/relative-entropy framework. Perlaza and Bisson study changes of expectation under changes of probability measure and their connections to Gibbs measures, information projections, and relative-entropy identities [PerlazaBisson2025]. This is prior context for the reweighting interpretation. The cosine-specific estimates and attainment argument needed here are proved explicitly above and below; this presentation does not claim to originate the Gibbs variational principle.

L.2. A rare-event construction

Suppose E has measure $p \in (0, 1)$, with $X \geq d > 0$ on E . For any $h \geq 0$, let

$$\lambda = \min\left(1, \frac{h}{\log(1/p)}\right), \rho = (1 - \lambda) + \lambda \frac{1_E}{p}.$$

Convexity gives $D(\rho) \leq \lambda \log(1/p) \leq h$. Since $E X = 0$, the objective is at least λd . Hence

$$b_A(h) \geq d \min\left(1, \frac{h}{\log(1/p)}\right).$$

Equation (L.6)

Applying (K.9) to consecutive frequencies gives, for $N \geq 2$,

$$b_{A_N}(h) \geq \frac{3(2N+1)}{20\pi} \min\left(1, \frac{h}{\log(3(2N+1)/2)}\right).$$

Equation (L.7)

Thus a fixed positive entropy budget can retain an order $N / \log N$ lower bound on this family, although fixed quantiles and the saturated detector lose the square-root scale. The construction adds only a small probability weight to the narrow deep event.

L.3. A universal upper bound and its direction

For every admissible A with $N \geq 2$,

$$b_A(h) \leq \left(h + \frac{1}{2}\right) \frac{N}{\log N}.$$

Equation (L.8)

To prove this, use the increasing function

$$j(u) = \frac{e^u - 1 - u}{u^2} = \int_0^1 (1-t) e^{tu} dt,$$

with $j(0) = 1/2$. Because $X \leq N$, for $\beta > 0$,

$$e^{\beta X} \leq 1 + \beta X + \frac{e^{\beta N} - 1 - \beta N}{N^2} X^2.$$

Taking expectations and using $\log(1+v) \leq v$ yields

$$g(\beta) \leq \frac{e^{\beta N} - 1 - \beta N}{2N}.$$

At $\beta = \log N / N$ this is less than $1/2$, and (L.5) gives (L.8). Together with (L.7), it proves $b_{A_N}(h) = \Theta_h(N / \log N)$ for every fixed $h > 0$.

The upper bound (L.8) holds for every set; the matching lower order has been proved only for consecutive frequencies. It would be an unjustified reversal to use (L.8) as evidence of a universal lower bound. The sought statement $b_A(h_0) \geq c\sqrt{N}$ with two absolute constants $h_0, c > 0$ remains unproved here.

L.4. Why variance alone cannot close the argument

For $N \geq 2$, consider a different random variable V that equals -1 with probability $N/(N+2)$ and $N/2$ with probability $2/(N+2)$. Direct arithmetic gives

$$EV = 0, EV^2 = N/2, |V| \leq N, -\min V = 1.$$

It has the first two moments and amplitude bound of the cosine model, but no square-root negative depth. It is not an exhibited cosine sum, so it is not a counterexample to Chowla. It shows exactly why an argument retaining only mean, variance, and a pointwise amplitude bound cannot suffice; additional integer-frequency structure is indispensable.

M. Frequency scaling and finite observation time

For every positive integer r ,

$$f_{rA}(x) = f_A(rx), K(rA) = K(A).$$

Equation (M.1)

The second identity holds because $x \mapsto rx$ maps the circle onto itself. Thus the gcd can be divided out without changing K . The raw largest frequency cannot by itself describe the negative depth: $\{1, 2\}$ and $\{r, 2r\}$ have the same value $9/8$, from

$$\cos x + \cos 2x = 2(\cos x + 1/4)^2 - 9/8.$$

Equation (M.2)

This invariance does not mean that the time required to observe a trough, or an approximation error based on a fixed time window, is independent of the frequency scale.

For positive real frequency vectors $\omega = (\omega_1, \dots, \omega_N)$, temporarily extend the model and define

$$K_T(\omega) = -\min_{0 \leq t \leq T} \sum_{i=1}^N \cos(\omega_i t), T \geq 0.$$

This quantity may be negative for a very short window; the definition is a signed minimum functional, not its positive part. For any two vectors ω, v ,

$$\left| K_T(\omega) - K_T(\nu) \right| \leq T \sum_i |\omega_i - \nu_i|.$$

Equation (M.3)

Indeed, cosine is 1-Lipschitz, so the two waveforms differ uniformly on the window by at most the right side; minima differ by no more than the uniform difference.

Over unbounded time, small frequency changes can behave differently. For irrational $\alpha > 0$,

$$-\inf_{t \in \mathbb{R}} (\cos t + \cos \alpha t) = 2.$$

Equation (M.4)

At odd multiples of π , the first term is -1 , while the irrational rotation allows the second to approach -1 . Here is the elementary density fact being used: among $m+1$ fractional parts $0, \alpha, \dots, m\alpha$, two lie within $1/m$; subtraction, and sign reversal if necessary, gives a nonzero orbit step $\theta \in (0, 1/m)$. Its multiples before passing one approximate every point of the unit interval within θ . Letting m grow proves density of the two-sided rotation orbit. A fixed translation of that dense orbit, as required for odd multiples, is still dense. Exact equality of both cosines to -1 would force α to be a ratio of odd integers, so an infimum rather than a minimum is needed.

Consequently, $K_\infty(1, 2+\eta) = 2$ for irrational η near zero, but $K_\infty(1, 2) = 9/8$. If a finite window achieves depth at least $2 - \delta$, where $0 < \delta < 7/8$, (M.3) forces

$$T \geq \frac{7/8 - \delta}{|\eta|}.$$

Equation (M.5)

A change in the infinite-time extremum may require an observation time diverging as the detuning shrinks. The integer-frequency problem, a real-frequency extension, and a finite-duration audio measurement are therefore distinct mathematical questions.

N. What has and has not been established

The proof architecture gives several independently checkable outcomes, but it does not remove the central missing estimate. The following distinctions should be retained in every public summary.

Statement: $1 \leq K(A) \leq N$ **Evidential status:** Direct proof, Proposition A.1 **Scope:** Every admissible set

Statement: The fixed-size infimum is attained **Evidential status:** Compactness proof, Theorem B.1 **Scope:** Each fixed N ; no effective exact-frequency cutoff

Statement: B_d is an attained moment optimization and tends to K **Evidential status:** Propositions C.1–C.2 **Scope:** Established pushforward hierarchy, specialized here

Statement: A degree in (E.1) ensures error at most ε **Evidential status:** Complete proof and rational sufficient certificate **Scope:** All admissible sets; no $\max A$ dependence

Statement: The lattice spectral edge equals $-K$ **Evidential status:** Exact Fourier equivalence **Scope:** Full infinite lattice

Statement: Open-box approximations converge with (G.3) **Evidential status:** Variational proof **Scope:** Fixed set; loss depends on distances

Statement: A grid with exact energy can miss a linear-depth trough **Evidential status:** Exact infinite family, Theorem H.1 **Scope:** Consecutive frequencies and the stated grid

Statement: Varying the boundary twist restores the continuous extremum **Evidential status:** Exact eigenvalue calculation **Scope:** Actual integer powers retained

Statement: Positive integer amplitudes do not guarantee the RMS-normalized extension **Evidential status:** Exact Fejér family **Scope:** Weighted extension, not the original conjecture

Statement: The Riesz detector's finite minimum is $7/20$ **Evidential status:** Exact finite enumeration **Scope:** Nonempty subsets of $\{1, \dots, 18\}$ only

Statement: Fixed negative moments below order two and fixed quantiles are insufficient **Evidential status:** Analytic interval-family obstructions **Scope:** The specific detectors

Statement: Every zero-threshold Fermi–Dirac parameter loses the scale on intervals **Evidential status:** Bound uniform in β , Proposition K.1 **Scope:** The zero-threshold family

Statement: Entropy optimization has exact duality and works at order $N / \log N$ on intervals **Evidential status:** Theorem L.1 and explicit bounds **Scope:** Lower order shown on that family, not all sets

Statement: $K(A) \geq c \sqrt{N}$ for a positive absolute c **Evidential status:** Not proved or disproved in this report **Scope:** The original open target

The archived programs provide exact coefficient arithmetic, finite comparisons, and checks of stated identities. Such checks are useful for detecting errors in constants, signs, and normalization. They do not replace a quantified proof over unrestricted integer frequencies. Conversely, a short proof for an infinite family can establish a genuine limitation of a method even when a numerical search over that family would be inconclusive.

The specialized uniform approximation theorem is presented as a checkable result of this project, with novelty unverified. An independent specialist should compare it with the existing pushforward and polynomial-concentration literature before it is described as a new research contribution. An internal verification pass is not external peer review, and publication of these materials does not imply endorsement by a journal or a mathematical institution.

The mathematical work also does not establish an audible improvement, a processing algorithm suitable for every audio signal, or a performance guarantee for Bass Plus. The connection to sound motivates the model and helps identify physically meaningful distinctions—amplitude versus squared amplitude, phase-sensitive fields versus intensities, continuous minima versus samples—but those distinctions do not expand the hypotheses of the original problem.

Citation keys used in this appendix

The publication package supplies full bibliographic entries. The core methodological references are:

[Lasserre2021] Jean B. Lasserre, *Connecting optimization with spectral analysis of tri-diagonal matrices*, *Mathematical Programming* 190, 795–809 (2021); DOI: 10.1007/s10107-020-01549-3; arXiv:1907.09784. Prior art for the pushforward hierarchy and its eigenvalue interpretation.

[SlotLaurent2021] Lucas Slot and Monique Laurent, *Near-optimal analysis of Lasserre's univariate measure-based bounds for multivariate polynomial optimization*, *Mathematical Programming* 188, 443–460 (2021); DOI: 10.1007/s10107-020-01586-y; arXiv:2001.11289. Prior analysis of the univariate measure-based bounds and their convergence.

[SlotLaurent2022] Lucas Slot and Monique Laurent, *Improved convergence analysis of Lasserre’s measure-based upper bounds for polynomial minimization on compact sets*, *Mathematical Programming* 193, 831–871 (2022); DOI: 10.1007/s10107-020-01468-3; arXiv:1905.08142. Prior art for Chebyshev and needle-polynomial concentration estimates.

[NISTDLMF] National Institute of Standards and Technology, *Digital Library of Mathematical Functions*, Section 18.5, “Explicit Representations,” equation 18.5.1. Source for the standard Chebyshev representation used in (E.3).

[PerlazaBisson2025] Samir M. Perlaza and Gaetan Bisson, *Variations on the Expectation Due to Changes in the Probability Measure*, *Entropy* 27(8), 865 (2025); DOI: 10.3390/e27080865. Context for changes of expectation, Gibbs measures, and relative entropy in Section L.

The reference dates in these keys refer to the bibliography’s publication identification; the linked preprints can have earlier submission dates. The historical investigations and source programs underlying this appendix are retained elsewhere in the package for reproducibility, rather than silently rewritten as if every exploratory statement had been correct from its first appearance.

Research Catalogue of Ninety Approaches

This catalogue records the actual scope of ninety numbered approaches developed in an AI-assisted investigation motivated by sound. The common question concerns

$$f_A(x) = \sum_{a \in A} \cos(ax), K(A) = -\min_x f_A(x), N = |A|,$$

where the frequencies are distinct positive integers, every amplitude is one, and all phases arise from the same variable. The desired conclusion is one absolute positive constant c such that $K(A) \geq c\sqrt{N}$ for every admissible set. **This collection neither proves that conclusion nor supplies a counterexample to it.**

The numbers identify research branches, not independent discoveries. Several branches revisit the same mechanism in a different language: frequency compression reappears as a finite-cutoff function; Gibbs weighting reappears as imaginary-time evolution; rational factorization reappears as amplitude quantization. Those connections are part of the record. A failed proposed inequality is an informative negative result only within its stated scope.

Evidence labels and reading conventions

Proved in the notes: a mathematical argument is supplied in the linked record. This label does not mean external peer review, formal proof-assistant verification, or established novelty.

Exact counterexample: an explicit construction disproves the specified intermediate assertion. It does not automatically disprove the original conjecture.

Finite computation: the conclusion applies to the reported finite search or certificate. Approximate reconnaissance and exact certification are distinguished.

Conditional or unresolved: a useful reduction has been made, but its sufficient hypothesis has not been established.

Reconstructed or historical: the original working files were lost. A replacement text or an earlier report survives; these are not interchangeable with the missing bytes.

The catalogue is written in English. Linked historical records retain their original English or Russian wording so that corrections and provenance remain inspectable. The accompanying verification report identifies what was rerun for this publication; references to older executions below are historical evidence, not a claim that every experiment has just been repeated.

Unnumbered groundwork: retained without retrospective numbering

Before the numbered series, the project explored exact finite searches, simulated annealing, Bernstein positivity certificates, spectral Gram matrices, damped products, scalar moments, and a probability-convolution reformulation. These materials are included as groundwork rather than being relabelled as additional approaches. Their main role is to establish the target, expose tempting relaxations, and supply reproducible small examples.

The finite-search directories include searches in bounded frequency universes, logs, and certificates. Such searches can certify statements inside their specified universes; none bounds the unrestricted maximum frequency. The general-method notes separately investigate whether the resulting patterns could support an all-

frequency theorem. A probability density with a flat nonzero Fourier spectrum gives a reversible reformulation, while a singular measure shows why absolute continuity matters. A damped-product monotonicity claim is refuted by an exact integer calculation. The moment and spectral records contain additional countermodels to restricted proposed estimates.

Sources: general checkpoint, original report, finite computation guide, finite-search audit, moment obstructions, and factorization notes.

A selected fresh reproduction during publication preparation rechecked the exact finite optimum over all 4,194,303 nonempty subsets of $\{1, \dots, 22\}$. Two different enumeration methods identified the same unique minimizer of $K(A)/\sqrt{|A|}$:

$$A_* = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 21\}.$$

Its depth is strictly between 2.4548327530 and 2.4548327531, certified by exact polynomial arguments. The finite comparison does not extend beyond the specified universe. Details and current execution evidence are in the groundwork reproduction report.

Approaches 01–20: unconventional mechanisms

01. Replace commuting rotations by free rotations

Outcome: exact transfer refuted; weaker comparison unresolved. The idea replaces the commuting shifts behind a cosine sum with noncommuting shifts, whose spectral edge is explicitly accessible. The proposed transfer of that edge back to the original one-dimensional model fails already for $A = \{1, 2\}$, where $K = 9/8$. A group quotient can compare the relevant positive edge without giving the required comparison of negative edges. The calculation therefore identifies a missing directional inequality, rather than invalidating spectral methods generally. A weaker comparison with an absolute loss remains a separate, unproved candidate.

Source: physics record.

02. Use fermionic determinants and exterior powers

Outcome: two specified shortcuts refuted. Positive Toeplitz matrices can be studied through determinants and exterior powers, suggesting a fermionic description of simultaneous constraints. The record checks the full matrices rather than assuming that a short Fourier support creates a short matrix rank. That rank-collapse premise is false. A further proposed inequality subtracting a variance contribution from a determinant quantity also fails on an explicit example. Positivity of all finite sections remains valid and useful, but it does not supply the missing term-count estimate without another argument linking those determinants to the negative spectral edge.

Source: physics record.

03. Locate complex-temperature partition-function zeros

Outcome: proposed zero-location premise refuted. The partition function $Z(\beta) = E e^{-\beta f_A}$ packages the full wave into a single analytic function. An analogy with Lee–Yang theory suggested that a rigid location of its complex zeros might force a large negative edge. For $A = \{1, 2\}$, the exact third moment contradicts the symmetry required by the proposed universal imaginary-axis zero argument. The physical theorem’s hypotheses cannot simply be imported into this cosine model. Partition functions remain a valid representation; later Gibbs branches pursue their real-temperature properties without the false zero-location premise.

Source: physics record.

04. Evolve the whole wave by heat flow

Outcome: evolution identities proved; proposed ratio control refuted. Heat evolution damps the mode a by $e^{-t a^2}$, allowing exact energy and minimum comparisons while treating the original sum as one function. However, this flow immediately replaces equal amplitudes with unequal amplitudes. An infinite evolved family defeats the proposed control relating its remaining energy to its negative minimum. The obstruction is to that particular quantitative transfer, not to heat analysis itself. Branch 85 revisits the same mechanism as phase jitter and supplies an explicit error budget that depends on the frequencies.

Source: physics record.

05. Stop a Brownian representation of the analytic polynomial

Outcome: scalar-data shortcut refuted. Evaluating the analytic polynomial along stopped Brownian motion gives a martingale representation and a quadratic-variation identity. The hope was that stopping-time information would force a negative excursion of order $\sqrt{N}$. An explicit stopped-process countermodel shares the proposed scalar constraints while violating the desired conclusion. Analyticity alone also fails to repair the tested inference. A successful refinement would have to retain additional information about the equal integer Fourier coefficients; mean, variance, and quadratic variation do not independently encode that structure.

Source: physics record.

06. Bias rational sampling with Gauss sums

Outcome: arithmetic partial bound proved; bounded-grid coverage refuted. Quadratic characters produce nonnegative sampling weights on roots of unity and an exact relation between residue information and a negative witness. This gives a valid arithmetic certificate when the frequency residues cooperate. Any predetermined finite collection of such grids, including grids combined through the Chinese remainder theorem, can nevertheless be defeated by an admissible family of frequencies. The missing step is a uniform choice of useful modulus. Failure of bounded preselected grids does not establish that adaptive rational sampling or all arithmetic certificates must fail.

Source: arithmetic record.

07. Force growth through Mahler measure and resultants

Outcome: sufficient lemma isolated but unproved. After an integer upward shift, the cosine polynomial has integer Laurent coefficients and is nonnegative on the unit circle. Its Mahler measure and resultants offer

arithmetic quantities that cannot vary arbitrarily. The notes formulate a lower bound strong enough to imply the target. Generic appeals to integrality do not establish it: the inspected reciprocal and cyclotomic examples show precisely where norms or resultants can be small or vanish. A result specific to the equal-coefficient spectrum would be needed. No new Mahler-measure theorem is claimed.

Source: arithmetic record.

08. Compress an entire frequency vector

Outcome: fixed-size attainment and effective approximation proved in the notes. The minimum is encoded by a circle subgroup of a torus. Compactness and the lattice of integer relations yield attainment of the extremal value for each fixed N . A separate quantitative argument preserves sufficiently many relations to replace any set by bounded frequencies while changing K by at most a prescribed error. The displayed bound is very large and depends on N and the error, but not on the original maximum frequency. This supplies an approximation procedure; it does not prove a lower bound uniform in N .

Sources: arithmetic record and separate audit.

09. Detune frequencies irrationally and take a limit

Outcome: interchange of limits refuted exactly. Infinitesimal irrational detuning can make phases explore a larger torus and permits near-simultaneous negative alignment. For $\{1, 2\}$, the original depth is $9/8$; for $\{1, 2 + \varepsilon\}$ with irrational ε , the unrestricted-time infimum has depth two. Returning the detuning to zero therefore does not recover the original full-time infimum. On any fixed observation interval the signals do converge. The obstruction is the order of the time and detuning limits, not continuity of cosine or the correctness of torus equidistribution.

Source: arithmetic record.

10. Read the complex image curve through area and winding

Outcome: identities valid; direct geometric inference refuted. The polynomial $P_A(z) = \sum z^a$ traces a complex curve whose signed area has an exact Fourier formula. Poisson extension also controls its values inside the unit disk. These are legitimate whole-function transformations, but signed area carries frequency weights that cannot be replaced automatically by the number of frequencies. Large exterior values similarly need not provide the desired boundary minimum. The tested transfers fail on explicit examples. Additional coefficient-sensitive geometry might still be useful; the recorded calculations do not eliminate that broader possibility.

Source: arithmetic record.

11. Round a positive Fejér taper with bounded prefix errors

Outcome: finite obstruction certified; asymptotic behavior unresolved. A prefix-floor rule converts a nonnegative weighted profile into an admissible zero-or-one frequency mask. Its cumulative rounding error is bounded, suggesting that positivity might survive approximately. Eight targeted sizes were checked. At $N = 131071$, an exact rational negative witness proves $K / \sqrt{N} > 241 / 125$ for the specified constructed set. This is a certified value at one point, not its global minimum. Neither that finite check nor the prefix-error identity rules out every possible asymptotic improvement or every other rounding rule.

Sources: construction record and stored checks.

12. Reverse the signal by a nonlinear change of time

Outcome: exact reversal and a uniform small-error variant refuted. The proposal seeks one transformation of the common time that maps the whole wave approximately to its negative. Such a transformation would transfer the obvious positive peak into a negative peak without independently altering the modes. Exact examples show that the necessary symmetry is absent. The notes also rule out the tested universal reversal with sublinear uniform error. Choosing a separate phase for every frequency would solve a different feasibility problem. Special parity-symmetric sets do possess exact reversal; branch 80 classifies their maximal-depth case.

Source: construction record.

13. Randomize whole sets and exchange the order of play

Outcome: relaxation too weak; domain correction preserved. Mixing entire frequency sets appears to invite a minimax proof. In the unrestricted frequency universe, the examined relaxed game has value $1/2$, so it cannot prove any lower bound growing with N . A later clarification distinguishes that universe from a fixed finite cutoff, where the exact value is different and explicitly calculated. This correction matters: an infinite-domain construction cannot be asserted for every bounded domain. The failed relaxation would require a rounding theorem preserving a negative witness, and no such theorem was obtained.

Sources: construction record and corrected game-domain statement.

14. Translate negative peaks into spin frustration and cuts

Outcome: valid transfer with an unresolved equivalent-scale requirement. A frequency distance graph supports a spin or cut interpretation. Properly constructed whole-graph cuts do yield witnesses for the original cosine problem. Independent-phase and magnetic-gauge shortcuts, however, do not preserve the one-parameter model and are explicitly rejected. The cut surplus sufficient for a square-root bound turns out to have essentially the same quantitative difficulty as the original target. This is a useful reformulation, not an independent proof. Later finite-graph and lattice-operator branches distinguish which graph compressions preserve enough information.

Source: construction record.

15. Clip negative values and restore the desired spectrum

Outcome: positivity shortcut refuted; iteration characterized. Clipping a wave makes it nonnegative, but projecting the clipped function back onto its original Fourier support can recreate negative values. An exact example demonstrates that failure. A natural alternating rule that adapts the constant term instead does converge to the unknown threshold $K(A)$. Its correctness as a recovery procedure is therefore compatible with its failure to prove any growth rate. A square-root theorem would need an additional bound on the total accumulated correction; convergence alone supplies no such quantitative estimate.

Source: construction record.

16. Seek a contradiction from a vanishing weak limit

Outcome: automatic energy contradiction refuted. A hypothetical sequence with unusually shallow negative peaks might converge weakly to a simple limit after normalization. The tempting inference is that its energy must then disappear, contradicting orthogonality. A concentrated weighted countermodel shows why

this is invalid: weak convergence need not preserve squared norms or prevent energy from concentrating on shrinking sets. The model is a test of the proposed compactness inference, not a set of unit-amplitude frequencies disproving Chowla. A successful limit argument would require an additional quantitative compactness or uniform-integrability property.

Source: information record.

17. Bound the information content of the shifted wave

Outcome: Shannon shortcut refuted; Rényi target equivalent to the problem. The normalized nonnegative function $1 + f_A / K$ is a probability density with an exact second moment. Its order-two Rényi entropy therefore encodes N / K^2 . A uniform bound of the required kind would indeed settle the target, but it is not independently established. The weaker suggestion that ordinary Shannon entropy controls that second moment fails because rare tall spikes can behave differently under the two functionals. This distinction motivates the later entropy-budget branch, where the exact variational problem and its scope are stated explicitly.

Source: information record.

18. Prevent hidden peaks with a sparse uncertainty inequality

Outcome: a valid but quantitatively insufficient bound. A Turán–Nazarov inequality controls an exponential polynomial on an interval from its values on a measurable subset. Applying it to the set where the negative contribution is limited yields a lower bound for K independent of the largest frequency. Its dependence on the number of terms is exponentially weak, however, and does not approach the desired square-root scale. The calculation is a legitimate application of an existing theorem. It rules out neither stronger uncertainty estimates for this special coefficient pattern nor other ways to exploit the same inequality.

Source: information record.

19. Replace trigonometry by an algebraic square certificate

Outcome: per-set certification valid; premature degree bound false. Substituting $y = \cos x$ gives $f_A = \sum T_a(y)$, so positivity can be certified on $[-1, 1]$ using standard polynomial sum-of-squares representations. This provides an exact method for a prescribed set. The polynomial degree is nevertheless tied to the largest frequency, even when the number of terms is small. A certificate-degree bound based on term count alone does not follow before compression. Branches 08 and 56 address approximate frequency compression, while branch 83 uses polynomials in the value of the wave instead.

Source: information record.

20. Use a passive-filter logarithmic balance law

Outcome: logarithmic identity valid; proposed shortcut refuted. Shifting the analytic polynomial into the positive-real half-plane creates a zero-free function and a mean-log identity reminiscent of a passive filter's balance law. The examined inference from this average logarithmic information to a sufficiently large negative excursion fails on an explicit zero-free family. Its true negative excursions can be linear while the proposed criterion does not capture them. This separates a correct analytic identity from an unsupported quantitative conclusion. It does not assert that passive-system methods, causal factorizations, or all logarithmic estimates are intrinsically unsuitable.

Source: information record.

Approaches 21–30: positivity, arithmetic, and discrete constructions

21. Divide the critical probability density into identical steps

Outcome: a general obstruction proved. The critical density $u = (K + f_A) / K$ is nonnegative and touches zero. If it were an integer convolution power of a probability measure, its finite Fourier support would force the root to have a nonnegative trigonometric-polynomial density. Convolving such a nonzero density with itself is strictly positive everywhere, contradicting the zero of u . Thus no identical probability root of order at least two exists. With an extra positive shift, the possible root order is still bounded by the largest Fourier coefficient. This structural fact alone has no growing dependence on N .

Source: analysis record.

22. Turn the shifted wave into a transport map

Outcome: exact reformulation; proposed continuity principle refuted. Integrating $1 + f_A / K$ gives a monotone degree-one map of the circle. Its derivative energy is exactly $N / (2K^2)$, so controlling that energy uniformly would prove the conjecture. Small displacement of the map does not supply this control: the primitive sets $\{q, q+2\}$ with odd q approach the identity uniformly while their derivative energy stays fixed. Even a one-frequency example has a non-Lipschitz inverse at a critical point. Transport geometry therefore preserves the question but does not answer it through these qualitative properties.

Source: analysis record.

23. Use only boundary contact points

Outcome: contact inequalities proved; required mass bound refuted. A Schur transform of the analytic polynomial produces exact angular derivatives and positive boundary Pick matrices at minimum points. These yield legitimate restrictions on contact data. The hoped-for lower bound for total contact mass in terms of N / K^2 fails for $A = \{1, D\}$, with odd D : there is one contact, its mass is $1 / (D+1)$, and $N / K^2 = 1 / 2$ stays fixed. The correction that a nonnegative zero has order at least two, not necessarily exactly two, is preserved. More coefficient information would be needed.

Sources: analysis record and audit.

24. Build a positive quadrature with one heavy atom

Outcome: transfer proved; uniform heavy-atom mechanism refuted. A positive quadrature annihilating the prescribed frequencies can turn a heavy atom into a lower bound on the negative excursion elsewhere. The difficulty is constructing that atom. For consecutive frequencies, a nonnegative squared polynomial forces every atom's weight to be at most $1 / (N+1)$, and therefore requires at least $N+1$ nodes. This defeats the tested universal square-root node count or heavy-atom claim. The obstruction is fully compatible with the existence of positive quadratures and does not rule out different sampling functionals with additional structure.

Source: geometry record.

25. Use every Galois conjugate at once

Outcome: trace identities proved; generic norm argument insufficient. Evaluating a strictly positive integer Laurent polynomial at a suitable root of unity gives positive algebraic-integer conjugates. Their sum, squared sum, and product have exact formulas. The product is a positive integer, but that fact does not control their relative variance. A cyclotomic unit family has norm exactly one while its second-moment-to-mean-square ratio diverges. Its coefficients are weighted, so it refutes a broader arithmetic shortcut rather than Chowla. Any successful strengthening must use the original flat nonzero Fourier coefficients essentially.

Source: geometry record.

26. Quantize coefficients of spectral factors

Outcome: conditional counting bound and an exact obstruction proved. Suppose $H = 2m + 2f_A$ is a sum of squared moduli with coefficients in $q^{-1}\mathbb{Z}[i]$. Counting coefficient energy gives the correctly normalized inequality $N \leq 2m^2 q^2 - m$. A uniformly bounded suitable denominator would yield the target. For $A = \{1, 2, 3, 5, 6, 7, 8\}$ at $m = 2$, exact positivity certificates coexist with an energy obstruction to denominator one. Rational decompositions still exist with some denominator for every strictly positive rational polynomial. The missing assertion is uniform denominator control, not existence of ordinary spectral factors.

Source: geometry record.

27. Remove signs from complementary codes

Outcome: the specified admissible family fails asymptotically. Complementary signed polynomials have a striking exact cancellation of squared moduli. The construction converts their signs into positive frequency masks in the hope of inheriting that cancellation with unit amplitudes. A continuous negative witness shows instead that the resulting family has $K / \sqrt{N}$ tending to infinity. An exact finite certificate also establishes $K > 6770$ at $N = 32895$. The asymptotic conclusion comes from a separate proof, not extrapolation of the computation. The result concerns the explicit sign-removal mechanism, not every use of complementary codes.

Sources: construction record and stored checks.

28. Cancel increasingly many frequency moments

Outcome: the specified Prouhet construction fails asymptotically. A Thue–Morse or Prouhet identity balances signed power sums through increasingly high orders. Removing the signs to form admissible frequency sets exposes an uncanceled dense background. An explicit phase then produces a negative excursion large enough that $K / \sqrt{N}$ diverges along the constructed family. At $N = 32768$, a rational certificate gives $K > 6951$. This illustrates why matching many frequency moments near one phase is not control of the complete cosine function. Other constructions with different density or geometry are not excluded.

Sources: construction record and stored checks.

29. Flatten positive Fourier multiplicities by fractional powers

Outcome: positivity preservation refuted for the tested exponent range. An autocorrelation polynomial is nonnegative but generally has repeated difference multiplicities. The proposal raises each positive coefficient to a fractional power, hoping to move continuously toward a flat support with equal coefficients. The record gives an explicit family for which the transformed polynomial has a negative value for every exponent $0 \leq p < 1$. Thus coefficientwise flattening does not preserve nonnegativity in this setting. This is an obstruction to the proposed operation; it neither contradicts spectral factorization nor proves that every alternative way to remove multiplicities must fail.

Source: construction record.

30. Forbid infinite strict descent at a fixed number of frequencies

Outcome: structural compactness result in the notes; no algorithmic stopping certificate. Extending branch 08, the record studies the set of attainable depths for a fixed N . Its subgroup-limit argument rules out an infinite strictly decreasing sequence of these values, hence gives a well-ordering property. It does not make the set of values finite; an explicit family distinguishes those assertions. Nor does it give an effective bound on the number of descent steps or a test that a current set is globally optimal. Most importantly, the statement is fixed-size and supplies no positive lower bound uniform over all N .

Sources: descent record and audit.

Approaches 31–40: quantitative certificates and controlled transformations

31. Solve coefficient equations modulo integers

Outcome: unrestricted congruences too weak; an energy-capped version exact. Reducing a sum-of-squares identity modulo an integer seems to permit local arithmetic tests before constructing a rational factorization. Without a coefficient-energy cap, the examined congruences can always be satisfied and provide no useful obstruction. With total energy E , a modulus larger than $E + q^2$ makes coefficient congruence imply exact equality. That is a valid finite arithmetic reduction, but enforcing the energy constraint retains the central difficulty. Passing modular equations alone should therefore never be reported as a rational positivity certificate.

Source: denominator record.

32. Construct a denominator independent of maximum frequency

Outcome: an explicit sufficient bound proved in the notes. With an additional positive shift, a binomial approximation to a square root is controlled in the Fourier coefficient norm. The number of frequencies in a power is bounded by the number of original modes and the power, not by their largest index. Rational coefficients and a diagonally dominant remainder then yield a finite sum of squares with a denominator depending on N and the positivity margin. The resulting bound is enormous. It proves existence constructively but does not give the absolute denominator needed for a square-root lower bound.

Sources: denominator record, audit, and power-norm audit.

33. Make a prescribed denominator a finite decision problem

Outcome: an existence bound for a finite search proved; full search not executed. If a decomposition exists at a specified denominator, its integral coefficient energy is fixed. The notes compress coefficient-position relations while keeping the original frequencies as anchors, producing an explicit finite degree bound for at least one representation. Hence denominator admissibility can in principle be decided by finite enumeration. The bound does not constrain every possible representation, since gratuitously shifted factors can have arbitrarily large degree. The full enumerative algorithm was not run; the code checks its component identities and examples rather than claiming a completed universal search.

Source: denominator record.

34. Replace positivity by all fractional inverse Hardy norms

Outcome: necessary inequalities proved; their joint converse refuted. Positive real part of $H = k + P_A$ implies $\| (k/H)^p \|_{H^2}^2 \leq \sec(\pi p)$ for every $0 < p < 1/2$. The hope was that retaining the whole family, rather than one norm, would recover the relevant positivity. For $H_N = 1 + z + \dots + z^N$, every inequality holds even when the trial shift does not make the real part nonnegative on the circle. The record proves a stronger norm estimate for this family. Thus all these averaged inverse norms together still do not replace the required pointwise condition.

Sources: analysis record and audit.

35. Cool the observation measure instead of changing the signal

Outcome: Gibbs identities proved; sufficient variance condition unresolved. Weighting time by $e^{-\beta f_A}$ favors negative values while leaving every original frequency and amplitude unchanged. Derivatives of the log partition function give the tilted mean and variance, yielding an exact candidate bridge to the desired scale. A proposed overly strong uniform variance lower bound fails on a certified small example. Direct comparison of densities gives only a weaker bound. The remaining variance statement is a concrete research target, not an established theorem. Branches 63, 66, and 82 develop the same weighting mechanism in other forms.

Source: analysis record.

36. Descend through fractional changes of frequency weights

Outcome: fractional descent proved; automatic integer rounding refuted. The directional derivative of the minimum depends exactly on the perturbation's values at the current contact points. This permits a strict fractional descent from every binary frequency set in the examined relaxation. An explicit example nevertheless has a negative initial derivative and a worse value after the corresponding full integer exchange. The descent direction therefore cannot simply be rounded into an admissible improved set. The result identifies a discrete barrier between the weighted relaxation and the original model; it does not imply that no multi-frequency exchange can improve a set.

Source: analysis record.

37. Correct cyclic aliasing before certifying the circle

Outcome: exact correction identity and continuous certificate proved. Cyclic autocorrelations combine differences modulo a period, whereas the original polynomial uses ordinary integer differences. The record writes the missing Laurent correction explicitly and then bounds the error between a sampled minimum and the continuous one by $\pi^2 \sum a^2 / (2Q^2)$ on a Q -point grid. This provides a valid continuous certificate once the sampled values themselves are controlled. The error depends on the frequencies, so a fixed grid is insufficient. Branch 90 later supplies a particularly transparent infinite family showing the same sampling danger.

Source: construction record.

38. Generate sparse frequencies with a Fibonacci automaton

Outcome: explicit sparse family fails as a counterexample candidate. To avoid the dense backgrounds that defeated branches 27 and 28, this construction uses digit strings with no adjacent ones. The admissible frequency set becomes sparse and its size follows a Fibonacci recurrence. A carefully chosen phase still produces a linear negative excursion: the record proves $K(A_m) > N_m / 20$ for every $m \geq 20$. Exact prefix probabilities support the witness. This is an infinite-family obstruction to the stated automaton construction, not evidence that sparsity itself must create a large negative minimum in every frequency set.

Source: construction record.

39. Encode a multidimensional torus by a random integer direction

Outcome: controlled probabilistic transfer proved in the notes. A trigonometric sum on a genuine torus can be restricted along an integer direction. Avoiding finitely many relation hyperplanes, with an explicit large sampling range, gives distinct positive one-dimensional frequencies whose depth differs from the torus minimum by at most a chosen error, with stated probability. This extends the relation-preservation mechanism of branch 08. An arbitrary coding direction need not work, and a finite torsion group is not interchangeable with a torus. The bound guarantees existence and probability, not computational efficiency at useful sizes.

Source: construction record.

40. Rotate one factor or dilate all frequencies

Outcome: arithmetic distinction and dilation invariance proved. Multiplying a spectral factor by a phase does not always make all its coefficients rational; an explicit positive one-frequency polynomial has no single Gaussian-rational factor at any denominator. A frequency dilation can change that single-factor question. In contrast, when any finite number of squared factors is allowed, the set of admissible denominators is invariant under the substitution $z \mapsto z^L$. Residue-class decomposition proves the reverse direction. This prevents common frequency scaling from removing the multi-factor denominator obstruction and clarifies which factorization problem is actually being studied.

Sources: phase and dilation record and audit.

Approaches 41–50: reconstructed interdisciplinary branches

The original working files for this round were lost after an environment failure. The linked texts are explicit reconstructions from surviving research statements. They contain inspectable arguments, but they are not

restored original files. In particular, the missing finite certificates in entry 42 must not be silently promoted to freshly verified evidence.

41. Synchronize the waves like coupled biological oscillators

Outcome: orbit-preservation failure proved in the reconstruction. A sinusoidally coupled oscillator system can synchronize phases, but its coupling vector generally leaves the one-dimensional orbit $(ax)_{a \in A}$. A Taylor expansion from zero initial phases proves this for distinct frequencies and nonzero coupling. Projecting the dynamics back onto that orbit merely changes the speed of traversal when the speed stays positive, leaving the minimum unchanged. The historical finite algebra checks are described as earlier checks, not recovered outputs. The mathematical obstruction is to this specific synchronization transfer; specially designed dynamics remain outside it.

Source: reconstructed biology record.

42. Search for an epistatic trap in single-frequency selection

Outcome: historical local-trap claim with a stated evidence gap. The earlier record reported that $\{1, 4, 5, 6\}$ is worse globally than $\{1, 2, 3, 4\}$, yet every single-frequency replacement worsens its depth beyond a fixed threshold. The infinite replacement tail, for the new frequency at least 500, has a complete reconstructed analytic proof. The 1,980 rational witnesses covering smaller replacements and several original positivity certificates were lost. Consequently the full local-minimum claim is retained as historical pending re-certification, not presented as newly reproduced. The intended lesson concerns interactions between simultaneous discrete changes, not a general asymptotic bound.

Source: reconstructed biology record.

43. Make the shifted wave a nonlinear population equilibrium

Outcome: exact finite-spectrum equilibrium ruled out in the stated model. A constant-coefficient linear differential operator preserves the largest frequency, while a homogeneous polynomial reaction of degree $p \geq 2$ generates a nonzero coefficient at p times that frequency. It cannot be cancelled in the proposed scalar equilibrium. The reconstruction also quantifies the unavoidable residual. Consecutive frequencies refute the particular C/N residual estimate that would have implied a square-root bound. Multicomponent systems, spatially varying coefficients, nonlinear derivatives, and infinite spectra are not addressed. The biological analogy motivates the question but does not validate the mathematical transfer.

Source: reconstructed biology record.

44. Use consensus and small individual influence

Outcome: transfer errors quantified; qualitative network shortcut refuted. A stochastic consensus matrix produces a weighted cosine average. Its distance from uniform influence gives exact uniform and squared-error bounds for transfer back to f_A . Correlations between the resulting opinions cannot be discarded: their sum retains the original variance. A rank-one reversible consensus network with individually small influences realizes a Fejér-weighted signal whose negative depth is constant despite growing variance. Thus rapid agreement and low influence do not ensure the desired excursion. A quantitatively accurate, informative consensus could still transfer a certificate, but its needed properties are unproved.

Source: reconstructed social record.

45. Infer global agreement from every proper coalition

Outcome: simultaneous phase-consistency shortcut refuted. An explicit integer frequency family has the property that each proper subset can nearly align negatively at some time, whereas the whole set cannot meet the stated common phase threshold. The threshold depends on the number of frequencies, a necessary qualification. This models the difference between locally compatible judgments and a globally consistent assignment. The same full frequency set nevertheless has a large negative sum, so near-unanimous alignment is stronger than the original task requires. The obstruction therefore refutes the coalition criterion, not the square-root conjecture.

Source: reconstructed social record.

46. Observe noisy votes generated by a common hidden phase

Outcome: exact Bayesian identities; ordinary-majority detector too weak. Conditionally independent binary votes are generated from the cosine values at one hidden time. Their unconditional higher correlations encode frequency relations, and their posterior mean is not automatically half the vote sum. The reconstruction proves a general square-root spread for the auxiliary votes. However, conditioning only on negative majority yields at most an $O(\log N)$ lower certificate on consecutive frequencies. That limitation concerns the detector's estimate, not the true depth. Rare, more informative voting events remain possible and motivate the later nonlinear product detector.

Source: reconstructed social record.

47. Spend polynomial degree more economically

Outcome: reconstructed improvement of a sufficient factorization degree. A Chebyshev approximation to the square root of a positively shifted polynomial is controlled on a Bernstein ellipse. Combining its approximation error with a support-count bound gives a sufficient degree of order $N^{3/2} \log(N+1)$ in the examined construction, improving the project's earlier bound. Rational rounding and exact remainder correction then produce suitable denominators. The denominator still grows rapidly with N , and the method does not establish the conjecture. This is a reconstruction of the argument; original historical parameter runs are not represented as recovered files or a literature-first result.

Source: reconstructed certificate record.

48. Add a monic helper and then try to remove it

Outcome: two precise facts, with a failed cancellation shortcut. A monic integer helper whose roots all lie on the unit circle can be cancelled from a sum-of-squares certificate without changing the denominator. A general strictly positive integer Laurent polynomial can also be multiplied by some monic helper to obtain a diagonally dominant integer certificate. The helper in the second statement need not satisfy the first statement's root condition. An explicit example exposes that distinction. Dividing it away gives rational functions of the variable, not necessarily polynomials with rational numeric coefficients. The apparent removal of the denominator obstruction is therefore invalid.

Source: reconstructed certificate record.

49. Combine two coprime denominators

Outcome: greatest-common-divisor gluing refuted; large-denominator gluing valid. The same positive polynomial has exact certificates at denominators 81 and 128 but none at denominator one. Their coprimality therefore does not permit a Bézout-style collapse of positive square sums. The certificates and impossibility argument are explicit, and the surviving integer checker in branch 50 verifies their arithmetic. A constructive positive statement goes in the other direction: sufficiently large multiples of the common divisor can be assembled from repeated certificates. The example uses a specific shift and does not automatically refute every differently shifted denominator conjecture.

Sources: reconstructed certificate record and surviving reconstructed checks.

50. Round a factor and repair the remainder exactly

Outcome: general rounding lemma and two exact certificates reconstructed and checked. If an approximate factor leaves a strictly diagonally dominant residual, sufficiently fine rational rounding preserves that margin. The residual then decomposes exactly into elementary squared factors. The notes give an explicit denominator threshold, valid for every larger integer, and a bound independent of the original maximum frequency. The 81 and 128 examples have surviving integer verification code and output. Earlier failed trial denominators are only historical failures of one rounding attempt; their missing vectors are not restored, and those failures do not prove denominator impossibility.

Sources: reconstructed rounding record, checker, and results.

Approaches 51–70: replacing a guessed constant by explicit functions

51. Define the exact size-dependent extremal coefficient

Outcome: quantifiers clarified; uniform positivity unresolved. Defining $c_N = \inf_{|A|=N} K(A) / \sqrt{N}$ makes precise the suggestion that the coefficient might be a function. The conjecture asks whether $\inf_N c_N > 0$. Each fixed-size value is positive, and the earlier attainment argument applies, but neither statement prevents those values from approaching zero as the size grows. The notes distinguish an estimate tending to zero from an actual counterexample sequence. Introducing the exact function is a useful organizational step; it is not a calculation of its values or a replacement for the missing uniform bound.

Source: extremal-function record.

52. Insert logarithmic or slowly varying corrections

Outcome: scale comparisons checked; stronger general bound not established. Candidates such as $\sqrt{N} / \log^p(N+2)$, iterated logarithms, and slowly varying corrections describe possible intermediate estimates. The branch checks what each proposed scale would and would not imply. A proved lower estimate that decays after normalization does not show that the true coefficient decays; that logical confusion would manufacture a false disproof. No universal lower bound of the proposed logarithmic strength is established here. These functions serve as explicit hypotheses and comparison targets, not fitted laws inferred from the available small examples.

Source: slow-correction record.

53. Hide the coefficient inside a nested loss recursion

Outcome: abstract criterion proved; link to the real extremal function missing. A recursive scheme might preserve a positive coefficient if each step loses only a fraction δ_n . For $0 \leq \delta_n < 1$, the product $\prod (1 - \delta_n)$ has a positive limit exactly when the losses are summable. Two telescoping models are checked exactly. This explains a necessary design principle for an iterative proof: indefinitely repeated small losses can still destroy the constant. No corresponding recursion for the actual c_N has been constructed, so the product criterion alone does not advance the conjecture's quantitative bound.

Source: recursion record.

54. Let the coefficient depend on frequency scale and greatest common divisor

Outcome: exact invariance and insufficiency examples. Multiplying every frequency by a positive integer reparametrizes the circle and leaves K unchanged. Thus a coefficient based only on raw maximum frequency confuses observation speed with amplitude geometry. Dividing out the common divisor removes that redundancy, but even N together with the primitive maximum frequency does not determine the minimum: $\{1, 3\}$ and $\{2, 3\}$ have different exact depths. More detailed arithmetic relations must be retained. This observation is compatible with useful bounds involving bandwidth; it only rejects those coarse statistics as exact descriptors.

Source: scale-and-gcd record.

55. Measure distance from a previously certified set

Outcome: a stable transfer bound proved; finite coverage absent. If two frequency sets differ in only a few members, their cosine sums differ uniformly by at most the size of their symmetric difference. Consequently $|K(A) - K(B)| \leq |A \triangle B|$, providing a simple exact way to transfer a certificate from a known set to nearby sets. The same result helps quantify robustness of a discrete search. A finite library of examples, however, does not cover all unbounded frequency sets within a useful small radius. The estimate becomes a universal theorem only with an additional covering argument, which is not supplied.

Source: set-distance record.

56. Replace unrestricted frequencies by a finite-cutoff function

Outcome: a fixed-size approximation procedure derived. The finite-universe extremal quantity decreases toward the unrestricted value as the cutoff increases. Branch 08's effective compression supplies an explicit cutoff for a prescribed absolute error, making this limiting statement quantitative at each fixed N . In principle one can approximate the unrestricted extremal coefficient by finite search with a certified error. The required range is huge and grows with N ; practical feasibility is not claimed. This is approximate computation of fixed-size quantities, not an effective universal stopping rule that proves one constant works simultaneously for every size.

Source: finite-cutoff record.

57. Use the average absolute signal

Outcome: general lower certificate proved; interval obstruction quantified. Since the cosine sum has mean zero, its positive and negative integrals agree. Therefore $K \geq \|f_A\|_1 / 2$. This is a directly interpretable whole-signal certificate based on average absolute amplitude. For consecutive frequencies, however, its value is only $O(\log N)$, while a direct Dirichlet-kernel witness gives a negative depth of order N . Average magnitude therefore misses the rare deepest events on this family. The conclusion concerns the effectiveness of this particular bound and leaves open combinations with more sensitive statistics.

Source: absolute-mean record.

58. Use fourth moments and additive energy

Outcome: a valid general estimate with a structural limitation. Hölder's inequality converts the exact second moment and the fourth moment into $K \geq N^{3/2} / (4\sqrt{2}\sqrt{M_4})$. Correct moment counting includes both relations of the form $a + b = c + d$ and those of the form $a + b + c = d$. The certificate can be strong when such relations are sparse, but consecutive frequencies have a sufficiently large fourth moment that the bound loses their true depth. Omitting one relation type would give an incorrect calculation. The method is therefore a tested partial certificate, not a universal square-root proof.

Source: fourth-moment record.

59. Let skewness determine the lower edge

Outcome: exact low-moment formula; two contrasting regimes. A two-dimensional moment calculation gives a lower bound from M_2 and M_3 . If there are no additive relations $a + b = c$, the third moment vanishes and the bound reaches $\sqrt{N/2}$. On consecutive frequencies, their many relations make the same bound tend to $2/3$, even as the true minimum grows linearly. This contrast shows what skewness captures and what it loses. The absence-of-relations case is a special class; its conclusion cannot be transferred to every spectrum by declaring the modes independent.

Source: third-moment record.

60. Optimize over all residue-count certificates

Outcome: exact arithmetic bounds; all-modulus limitation proved. Averaging the cosine sum on roots of unity expresses a certificate through divisibility counts. The branch treats the modulus as a variable and asks whether choosing it optimally resolves earlier fixed-grid failures. For $A = \{1, \dots, N\}$, the maximum of this particular bound over every modulus is exactly one. Thus allowing all moduli does not repair this restricted averaging statistic. This is stronger than failure of one chosen grid, but it does not concern all possible weighted arithmetic samples or continuous interpolation between the nodes.

Source: residue-function record.

61. Read the spectrum of a frequency graph

Outcome: a graph certificate proved; examples separate it from scalar moments. The distance graph on the specified frequency vertices gives $K \geq -\lambda_{\min}(G_A)/2$. On consecutive frequencies the graph becomes complete, so this finite compression loses their linear negative excursion. A different constructed mixed family shows that graph information can nevertheless produce square-root growth while the third-moment certificate remains bounded. The two statistics therefore preserve different structure. Neither dominates universally in the tested argument. The exact infinite-lattice operator in branch 78 retains the full minimum but does not make its estimation automatic.

Source: graph-function record.

62. Increase the degree of a moment hierarchy

Outcome: monotone convergence and uniform approximation in the notes. The Rayleigh-type quantities $B_d = -\min_{deg p \leq d} E[f_A p(f_A)^2] / E[p(f_A)^2]$ increase toward K . A near-minimum measure estimate yields a sufficient degree depending only on N and the desired error, without bounding the maximum frequency. On intervals, degrees one and two have bounded limiting values, while an explicit cubic filter already gives a linear certificate. This does not improve the known direct interval witness. The unresolved step is a square-root lower estimate for general sets; branch 83 improves the sufficient approximation degree.

Source: moment-hierarchy record.

63. Make temperature a variable certificate

Outcome: exact convergence; universal rate target unresolved. For positive β , $\log E e^{-\beta f_A} / \beta$ increases toward the negative depth. Temperature thus turns a guessed coefficient into an explicit function of the complete signal. This restates the Gibbs mechanism without altering any wave amplitudes. The missing result is a choice of temperature that gives a square-root lower bound uniformly for every spectrum. Countermodels show that the first two scalar moments alone cannot supply it. Convergence as temperature tends to zero in physical terminology should not be mistaken for a quantitative uniform growth theorem.

Source: Gibbs-function record.

64. Vary the order of the negative-part moment

Outcome: exact asymptotic regimes on intervals; general quadratic case open. Let $f_- = \max(-f_A, 0)$. Its L^p size increases toward K as p increases. For consecutive frequencies, the record establishes bounded behavior for $p < 1$, logarithmic growth for $p = 1$, and order $N^{1-1/p}$ for $p > 1$. Hence every fixed $p < 2$ is too weak to certify a universal square-root scale through this statistic. The interval calculation does not refute the candidate at $p = 2$, nor does it prove it for general sets. The distribution's negative tail, not total variance alone, is decisive.

Source: negative-moment record.

65. Ask how often a deep negative value occurs

Outcome: quantile certificate valid; fixed-probability requirement too strong. A negative quantile supplies a lower bound for K with an interpretable sampling probability. The proposal would be especially useful if a square-root-depth event occupied a fixed positive fraction of every period. Consecutive frequencies rule that out: the measure of such a negative event is at most order $N^{-1/2}$. Their deepest excursions remain large but rare. Therefore fixed-frequency random sampling can systematically miss the relevant region. This is a limit on the requested probability guarantee, not an upper bound on the true minimum's depth.

Source: quantile-function record.

66. Budget the information needed to favor rare times

Outcome: variational formula proved; a useful special-family estimate. The branch maximizes the negative weighted mean while bounding relative entropy of the observation density. The resulting variational formula precisely quantifies the cost of concentrating attention on rare negative events. For every fixed positive entropy budget, consecutive frequencies admit a certificate of order $N / \log N$, overcoming the fixed-quantile obstruction on that family. No analogous square-root lower bound has been proved for all sets. This revisits branches 17 and 35 with a controlled optimization problem; it does not identify an actual thermodynamic entropy of an audio device.

Source: entropy-budget record.

67. Treat the denominator itself as the unknown function

Outcome: corrected conditional estimate; shift-dependent counterexamples distinguished. For an admissible factor denominator q and $m = \lfloor K \rfloor + 2$, the correct coefficient-energy count is $N \leq 2m^2 q^2 - m$, yielding $K \geq \sqrt{N} / (3\sqrt{2}q)$. A denominator bounded uniformly would suffice, but no such bound is constructed. The earlier obstruction at shift $m = 2$ must not be transferred to shift $m = 3$: the same seven-frequency example has an explicit denominator-one certificate at the latter shift. The catalogue uses the corrected q^2 normalization and preserves the earlier error in the historical record.

Source: denominator-function record.

68. Include resonance and observation duration in the function

Outcome: exact discontinuity and finite-time continuity distinguished. This returns to the detuning example of branch 09 to make its practical meaning explicit. The full-time infimum jumps when $\{1, 2\}$ is replaced by $\{1, 2 + \varepsilon\}$ with irrational detuning, yet a fixed observation interval admits an error bound proportional to its duration and the detuning. Resonance information therefore cannot be replaced by merely close numeric frequencies when unlimited time is allowed. The noninteger example is a test of an inference, not a change to Chowla's domain. It also does not assert that a finite audio recording observes the unrestricted infimum.

Source: resonance-function record.

69. Take the strongest of several available functions

Outcome: valid combination; specified envelope remains insufficient. Taking the maximum of lower certificates preserves validity and can improve individual examples. The branch combines six previously checked estimates and asks whether their weaknesses complement one another. Consecutive frequencies show that all six can remain weak together: their maximum is still only $O(\log N)$, despite a true linear depth. This does not rule out every possible envelope or adaptive certificate selection. In fact, adding the cubic moment filter of branch 62 changes the conclusion on this family. The obstruction is to the documented finite menu, not to combination as a principle.

Source: function-envelope record.

70. Replace frequencies by their smooth distribution

Outcome: an exact observation-window obstruction. The constructed distinct integer frequencies approximate a smooth profile whose cosine transform is nonnegative. At scaled times $x = s / (2N)$, the deviation from that limiting profile is at most $|s|$. Consequently a window $|s| \leq T$ cannot certify a negative depth exceeding T . For $T = o(\sqrt{N})$, this defeats a square-root certificate based only on that window. The full signal outside the window is not controlled by the conclusion, and no small global K is proved. Smooth frequency statistics can therefore conceal the scale where a decisive excursion occurs.

Source: frequency-profile record.

Approaches 71–90: physical meaning and quantum analogies

71. Interpret the coefficient as a negative crest factor

Outcome: an exact physical normalization. For a pressure-like signal $p(t) = P_0 f_A(\omega_0 t)$, its full-period RMS is $P_0 \sqrt{N/2}$ and its negative peak depth is $P_0 K$. Their ratio is $\sqrt{2} K / \sqrt{N}$, making the candidate coefficient a dimensionless shape statistic. Changing units or scaling all frequencies does not determine it. Equal RMS can coexist with different depths. Negative pressure perturbation describes rarefaction relative to equilibrium, not negative energy. The interpretation clarifies what is being sought without making the coefficient a material constant or a quantum of energy.

Source: crest-factor record.

72. Compare quantum return amplitude with return probability

Outcome: amplitude representation exact; probability-only inference refuted. An equal superposition of frequency-labelled basis states gives a return amplitude whose real part is f_A / N . The survival probability uses its squared modulus and loses a global phase. Shifting every frequency by a common carrier changes that phase while preserving the probability curve. The sets $\{1, 2\}$ and $\{2, 3\}$ consequently have identical return probabilities but different exact negative depths. Thus probability alone cannot recover the target. Branch 86 uses the same counterexample in classical interference language; these are related interpretations, not two independent discoveries.

Source: quantum-amplitude record.

73. Use a quantum speed limit as a clock

Outcome: necessary time bound; no existence theorem. The elementary cosine inequality gives a lower bound on the real return amplitude in terms of the frequency second moment and elapsed time. It therefore supplies a necessary time before a specified negative threshold can be reached. It does not prove that the threshold is ever reached. Common frequency dilation changes that clock while leaving K unchanged. A negative real amplitude also does not imply orthogonality: a single basis state has survival probability one while its phase rotates. These distinctions prevent the speed-limit analogy from manufacturing a depth estimate.

Source: quantum-clock record.

74. Decohere the modes into independent phases

Outcome: exact distributional transfer refuted. Independent uniform phases give a symmetric sum and a negative-part second moment of $N/4$. The original integer-frequency curve has locked phases and need not share that statistic. For $\{1, 2\}$, direct integration gives $E f_-^2 = 2/3 - 9\sqrt{3}/(16\pi) < 1/2$, unlike the independent model. The numerical difference is a rigorous counterexample to the proposed equality or universal transfer. It does not show that the original sum has a small minimum. Random phase changes may be useful experimentally, but they alter the mathematical object under investigation.

Source: decoherence record.

75. Count phonons instead of counting frequencies

Outcome: the quantities are mathematically distinct. Quantizing a harmonic oscillator produces number states with nonzero excitation energy but zero mean displacement. A coherent state can instead have a classical sinusoidal mean. Applying the latter description to all modes returns the original cosine problem, while integer occupation numbers do not enforce its unit mean amplitudes. The branch therefore checks the quantum analogy through exact expectation and variance formulas. It neither denies phonons nor proposes a new quantum law. The number of frequencies, the number of excitations, the amplitude, and the energy cannot be substituted for one another in a proof.

Source: phonon record.

76. Squeeze quantum uncertainty to force a classical excursion

Outcome: mean-versus-noise confusion refuted. Displacement changes a quantum state's mean quadrature without fixing its uncertainty, while squeezing changes the uncertainty distribution. Thus the same mean sinusoidal signal can coexist with different quantum measurement noise. Heisenberg-type variance constraints concern repeated measurements of operators, whereas $N/2$ in this problem is the time average of the squared classical mean signal. Equating those variances is unjustified. The calculations identify why uncertainty alone does not impose the desired negative classical peak. They do not exclude other operator inequalities that explicitly preserve the relevant mean waveform.

Source: squeezing record.

77. Read the square-root scale as a count of interference pairs

Outcome: exact identity and an upper bound, not the desired lower bound. If a set of m emitters has distinct positive pair differences, those differences form $N = m(m-1)/2$ modes and satisfy

$\left| \sum_b e^{ibx} \right|^2 = m + 2f_A(x)$. Nonnegative intensity gives $K \leq m/2$. A visibility function records when that upper value is approached. The square-root relation here comes from pair counting, not from a universal quantum amplitude unit. This known construction explains why square-root scales are natural, while its inequality points in the opposite direction from the conjecture's required bound.

Source: interference-pair record.

78. Make the cosine sum a particle's lattice dispersion

Outcome: exact full-spectrum correspondence. On the infinite integer lattice, let T be the unit shift and $H_A = 1/2 \sum_{a \in A} (T^a + T^{-a})$. Fourier transformation makes this operator multiplication by f_A , so its spectrum is exactly $[-K, N]$. Frequencies become allowed hopping distances, and the negative minimum becomes the lower spectral edge. This retains the entire original set. A discrete spatial lattice does not imply a discrete energy spectrum on an infinite system. The reformulation creates useful trial-state tools, but a universal low-energy state of the required depth remains to be found.

Source: lattice-operator record.

79. Put the particle in a finite open box

Outcome: rigorous convergence with a frequency-dependent error. Compressing the lattice operator to a box of length L yields a computable lower certificate $B_L \leq K$. A truncated plane-wave trial state proves $0 \leq K - B_L \leq \sum_a \min(a/L, 1)$, and the certificates increase toward K . This is a genuine approximation theorem. A box length depending only on the number of modes can miss all relevant hops: one frequency larger than the box gives a zero compressed operator despite depth one. Boundary choice and frequency scale must therefore be included in finite-matrix approximations.

Source: finite-box record.

80. Classify maximal depth through bipartite symmetry

Outcome: exact characterization of a special equality case. The maximum possible negative depth is N . It occurs exactly when, after division by the common greatest divisor, all frequencies are odd. At the corresponding phase every cosine equals minus one. The same criterion makes the lattice graph bipartite and produces a chiral spectral symmetry. Conversely, simultaneous negative alignment enforces that arithmetic parity pattern. This completely resolves the maximal-depth equality case, not the general lower bound when odd relations frustrate alignment. It also clarifies the special circumstances under which the time-reversal proposal of branch 12 works.

Source: bipartite-symmetry record.

81. Quantify frustration on an odd cycle

Outcome: a proved upper restriction on attainable depth. An odd integer relation among the allowed hops prevents all the associated phases from being simultaneously opposite. Applying the odd-cycle spectral bound gives $K \leq N - r(1 - \cos(\pi/r)) / \max |k_a|$, with $r = \sum |k_a|$ for the specified odd relation. This quantifies the loss from the ideal depth N . Its direction is crucial: it is an upper bound on depth, not a lower certificate approaching Chowla's goal. The result helps interpret structural frustration but cannot be quoted as progress toward the desired inequality merely because it contains a nontrivial gap.

Source: odd-cycle record.

82. Evolve the lattice particle in imaginary time

Outcome: exact Gibbs correspondence; sign shortcut blocked. The diagonal matrix element $\langle 0 | e^{-\beta H_\Lambda} | 0 \rangle$ equals the cosine sum's partition function. Its power series counts closed lattice walks. Even and odd walk lengths enter with different signs; odd return walks are present when the graph lacks the parity symmetry of branch 80. Discarding their negative contributions cannot establish the proposed lower estimate. The representation may help organize the exact integer relations, but it does not eliminate them. This is the operator version of the Gibbs branches, not an additional independently discovered statistical law.

Source: imaginary-time record.

83. Amplify the lower edge with a Chebyshev filter

Outcome: improved sufficient approximation degree proved and separately checked internally. A near-minimum phase-cell estimate supplies a frequency-independent amount of measure. A Chebyshev polynomial suppresses the rest of the spectral interval while amplifying that region. Combining the two gives $0 \leq K - B_d \leq \varepsilon$ with an explicit sufficient degree of order $N^{3/2} \varepsilon^{-1/2} \log(2N/\varepsilon)$ in the stated parameter range. This improves the project's earlier sufficient bound, not a demonstrated best published algorithm. The unknown minimum enters the existence proof's trial polynomial; the moment optimization itself does not require it as input. The conjectured depth is still unproved.

Sources: Chebyshev-filter record and internal independent-pass audit.

84. Bound the duration of a negative wave packet

Outcome: curvature gives a rigorous local-width certificate. At a minimum, the derivative vanishes and the second derivative has magnitude bounded by $\sum a^2$. Taylor's inequality therefore gives a neighborhood staying below half the minimum depth, and normalized measure at least $\pi^{-1} \sqrt{K / \sum a^2}$. This connects depth, curvature, and the duration of one negative excursion. Common frequency dilation narrows each individual feature without changing K , while it also repeats features more often. The record distinguishes one feature's width from total negative-set measure and does not identify this classical estimate with quantum uncertainty.

Source: wave-packet record.

85. Interpret smoothing as common phase jitter

Outcome: exact error control; fixed damping fails uniformly. Heat smoothing multiplies the mode amplitudes by $e^{-\tau a^2}$ and cannot increase the negative depth. The bound $0 \leq K - K_\tau \leq \tau \sum a^2$ allows a chosen approximation error if the jitter level adapts to the spectrum. At any fixed positive τ , primitive sets such as $\{M, M+1\}$ are almost erased as M grows, although their original depth remains at least one. Thus fixed smoothing cannot provide a universal certificate. This develops branch 04 with a physical interpretation and an explicit frequency-dependent tolerance.

Source: heat-damping record.

86. Recover the sign with an optical-style reference wave

Outcome: exact differential-intensity identity. Measuring only $|S_A|^2$ loses the phase needed to determine $\operatorname{Re} S_A = f_A$. Adding a real reference R at two opposite signs yields $|R + S_A|^2 - |R - S_A|^2 = 4R \operatorname{Re} S_A$. This shows how a signed waveform can be recovered from positive intensity measurements. The identical-intensity, different-minimum example is the same carrier-shift obstruction used in branch 72. The method restores information; it does not manufacture a stronger negative excursion. No actual Bass Plus measurement implementation or quantum experiment is claimed in this mathematical record.

Source: reference-wave record.

87. Multiply destructive-interference weights

Outcome: exact detector; trial constants refuted and finite search retained. The positive product $\prod_{a \in A} (1 - \cos ax)$, normalized to a probability density, gives a lower certificate computable from integer Laurent coefficients. For powers of four it equals $N/2$. The proposed universal coefficient $1/2$ is false; an explicit nine-frequency example has normalized certificate $7/20$. The historical complete search over 262,143 nonempty subsets of $\{1, \dots, 18\}$ has the same minimum. This is a finite-domain statement. A positive lower coefficient for all sets is unresolved; even general nonnegativity of the raw detector is not separately proved.

Sources: Riesz-product record and stored exact finite results.

88. Favor low values with Fermi–Dirac occupation

Outcome: valid weighting; zero-threshold detector uniformly too weak on intervals. The decreasing weight $1/(1 + e^{\beta f_A})$ produces a nonnegative lower certificate after normalization. Its saturation, however, prevents it from concentrating arbitrarily strongly on rare deep events. For consecutive frequencies the record proves an upper bound $20 + 10 \log(2N + 1)$ for that certificate simultaneously for all nonnegative β . Consequently no temperature choice rescues the specified zero-chemical-threshold detector at the square-root scale. Moving the threshold defines a different problem not settled by this argument. The Fermi–Dirac analogy does not change the original signal’s amplitudes or require an actual fermion system.

Source: Fermi–Dirac record.

89. Permit amplitudes to come in integer quanta

Outcome: broader weighted conjecture refuted by an infinite family. Equal amplitudes are stronger than merely integer amplitudes. The triangular weights $N, N-1, \dots, 1$ give a scaled Fejér polynomial whose negative depth divided by $\sqrt{\sum w_a^2}$ tends to zero. The corresponding unscaled positive taper has constant negative depth. Thus amplitude quantization alone cannot replace the equal-unit-coefficient assumption. This is an exact counterexample to a weighted extension, not to Chowla's original formulation. It also explains why an equalizer's arbitrary gain settings cannot inherit a mathematical guarantee proved only for equal-amplitude modes without another transfer argument.

Source: amplitude-quantization record.

90. Quantize phase on a ring and then twist the boundary

Outcome: exact sampling counterexample and complete boundary-phase recovery. For $A = \{1, \dots, N\}$, sampling at $2N+1$ equally spaced phases gives depth exactly $1/2$, while the continuous depth is at least $1/2 + (2N+1)/(3\pi)$. The grid retains the exact mean square and enough information for interpolation, but its smallest sample is not the continuous minimum. On a finite particle ring, these samples are eigenvalues under one boundary condition. Varying the boundary phase fills the continuous dispersion and recovers its minimum exactly. Quantization therefore requires boundary information; it does not supply an automatic universal constant.

Sources: phase-grid and boundary-twist record and separate internal audit.

What the catalogue establishes as a collection

The useful output is a set of explicit reductions, approximation guarantees, conditional certificates, and counterexamples to identified shortcuts. The strongest quantitative approximation result here is branch 83; it controls how accurately a moment hierarchy can approximate a minimum independently of the original maximum frequency. It does not control the minimum's required growth with N . The source comparisons in the main article determine how these observations relate to existing literature; a large catalogue by itself establishes neither novelty nor journal suitability.

The direct physical interpretation in branch 71, the lattice correspondence in branch 78, and the sampling example in branch 90 support accurate educational explanations. They do not demonstrate improved sound quality, a new acoustic effect, or a deployed product feature. The historical checks are valuable because their inputs, exact claims, limits, and occasional missing files are visible. A future extension should add a dated record and preserve this boundary between a proved special statement, a finite test, and a still-open universal claim.

References and Literature Context

This bibliography supports the curated research report and its companion pages. Sources were checked on 18 October 2026. The descriptions identify why a reference is relevant; they do not claim an exhaustive priority search or an independent verification of every result in the cited literature. Attribution is provided through authors, titles, years, and publication identifiers, without outgoing website links. The cited external papers are not redistributed in this archive. Historical source notes retain their original form and are supplied as research records.

Chowla’s problem and recent progress

JMTZ2025

Zhihan Jin, Aleksa Milojević, István Tomon, and Shengtong Zhang. **From small eigenvalues to large cuts, and Chowla’s cosine problem.** arXiv:2509.03490, version 2, 27 October 2025.

The paper connects spectral graph methods to polynomial lower bounds for the negative minimum of cosine sums. It provides context for the gap between available polynomial bounds and the square-root target. The present report does not reprove its graph-theoretic results.

Bedert2026

Benjamin Bedert. **Polynomial bounds for the Chowla Cosine Problem.** arXiv:2509.05260, version 3, 24 July 2026; first submitted in 2025.

The version checked reports a lower bound of order $N^{1/5-o(1)}$. Older cached summaries of the first version state a different exponent. The version number matters when describing the mathematical landscape.

Shankar2026

Abhishek Shankar. **A Log-Free $n^{1/5}$ Bound for Chowla’s Cosine Problem.** arXiv:2609.05338, 4 September 2026.

This preprint reports removal of a subpolynomial loss from a related lower bound. It is cited as recent literature, with its preprint status retained. Its proof is not used as a verification certificate for any computation in this package.

BuEtAl2026

Bryan Bu, Fayzan Khan, Hyunjun Yi, Atharv Jha, Vihaan Vajpeyi, Ishaan Chandupatla, and Madhav Nagpal. **A Half-Constant Upper Bound for Chowla’s Cosine Problem.** arXiv:2610.17160, 12 October 2026.

This very recent preprint concerns constructions giving upper bounds on the extremal function. It is included in the literature watch because construction upper bounds and universal lower bounds play different roles. No comparison in this report is presented as an independent validation of that preprint or a claim of priority over it.

Established optimization methods

Lasserre2021

Jean B. Lasserre. **Connecting optimization with spectral analysis of tri-diagonal matrices.** *Mathematical Programming* 190, 795–809 (2021), published online 8 August 2020. DOI: 10.1007/s10107-020-01549-3. Author preprint, arXiv:1907.09784.

This is the direct prior art for the univariate pushforward measure hierarchy used in the report. The general approximation mechanism, its eigenvalue interpretation, and basic convergence are established methods. The archive’s polynomial quotient is a specialization of this framework, not a new hierarchy.

SlotLaurent2021

Lucas Slot and Monique Laurent. **Near-optimal analysis of Lasserre’s univariate measure-based bounds for multivariate polynomial optimization.** *Mathematical Programming* 188, 443–460 (2021), published online 30 October 2020. DOI: 10.1007/s10107-020-01586-y. Author preprint, arXiv:2001.11289.

The paper studies convergence of the same pushforward approach using polynomial concentration and estimates for mass near a minimum. The present report’s explicit degree bound is compared with an earlier bound from this project, not claimed to improve the best general result in this literature. A separate priority analysis would be needed for the specialization that is uniform in the largest frequency.

SlotLaurent2022

Lucas Slot and Monique Laurent. **Improved convergence analysis of Lasserre’s measure-based upper bounds for polynomial minimization on compact sets.** *Mathematical Programming* 193, 831–871 (2022), published online 25 January 2020. DOI: 10.1007/s10107-020-01468-3. Author preprint, arXiv:1905.08142.

Its Chebyshev and needle-polynomial estimates are relevant prior art for the filtering step. Journal issue dates, online publication dates, and preprint dates are different and should not be conflated.

NISTDLMF

NIST Digital Library of Mathematical Functions. **Section 18.5, Explicit Representations; equation 18.5.1.**

Used for the standard representation of Chebyshev polynomials, including $T_d(\cos \theta) = \cos(d\theta)$ and its continuation outside the unit interval.

PerlazaBisson2025

Samir M. Perlaza and Gaetan Bisson. **Variations on the Expectation Due to Changes in the Probability Measure.** *Entropy* 27(8), 865 (2025), published 14 August 2025. DOI: 10.3390/e27080865.

The paper connects changes of expectation to Gibbs measures, information projections, and relative-entropy identities. It supplies context for the observation-measure interpretation in Appendix L. The cosine-specific estimates are proved in that appendix; the underlying Gibbs framework is established prior work.

Physical interpretation

BakerAcoustics

Richard Baker, University of Manchester. **Basic Acoustics, Chapter 3: Sound Intensity and Pressure.** University teaching resource.

Supports the distinction between acoustic pressure and intensity, with the impedance relation under the plane progressive-wave assumptions. The report's ideal cosine model does not substitute for a measured acoustic system.

TongAQM

David Tong, University of Cambridge. **Lectures on Applications of Quantum Mechanics, Chapter 3: Electrons in Solids.**

Context for tight-binding operators, dispersion relations, and spectral bands on a lattice. The Fourier identities in the appendix are proved directly.

Bardyn2014

Charles-Edouard Bardyn, Sebastian D. Huber, and Oded Zilberberg. **Seeing bulk topological properties of band insulators in small photonic lattices.** *New Journal of Physics* 16, 123013 (2014). DOI: 10.1088/1367-2630/16/12/123013. Author preprint, arXiv:1312.6894.

An example of established use of twisted boundary conditions. The report does not claim to invent this spectral technique or to obtain a topological result about audio.

Cappellaro2012

Paola Cappellaro. **22.51 Quantum Theory of Radiation Interactions, Complete Course Notes.** MIT OpenCourseWare, Fall 2012; especially the harmonic-oscillator, coherent-state, and phonon sections. Official course notes.

Provides the physical vocabulary for number states, coherent states, and phonons. The report keeps mean displacement, variance, and number-operator eigenvalues distinct. Number states with nonzero occupation should not all be described as minimum-uncertainty states.

Shapiro6453

Jeffrey H. Shapiro. **6.453 Quantum Optical Communication.** MIT OpenCourseWare, Fall 2016. Selected official readings:

Reading 4: quantum harmonic oscillator and number-state formalism.

Reading 5: coherent-state properties.

Reading 6: quadrature noise and squeezed states.

These readings support the physical distinctions used in the companion essay. A classical Fourier coefficient is not forced to be an integer by the number spectrum of a quantum oscillator.

OConnell2010

A. D. O’Connell and coauthors. **Quantum ground state and single-phonon control of a mechanical resonator**. *Nature* 464, 697–703 (2010). DOI: 10.1038/nature08967. Research-group-hosted paper and supplementary material.

An experimental reference establishing the physical context of single mechanical quanta. It supplies no direct theorem about the unit-amplitude cosine sums studied here.

Formula rendering for the website**KaTeXDocs**

KaTeX official documentation: Node.js rendering and rendering options. The implementation handoff recommends build-time rendering with output: "htmlAndMathml", local CSS and fonts, and a failed build for unsupported formulas rather than silent raw-TeX output.

MathJaxDocs

MathJax official documentation: What is MathJax?. It is an alternative if the existing site already uses MathJax or an expression cannot be handled by the selected KaTeX pipeline. Running both renderers on the same expressions is unnecessary.

Scope of the literature comparison

The principal attribution correction is explicit: the moment hierarchy is Lasserre’s, and the Chebyshev concentration mechanism has established precedents. The appendix’s frequency-independent sufficient degree is supplied with a full proof, but no claim of scientific priority. This package reports its own verification status separately from the publication status of external sources. A future journal submission would require an expert assessment of novelty, positioning, and the proofs.

Reproducing the Research Report

Release: 18 October 2026. The archive contains the English publication, exact-arithmetic code, preserved historical material, and records of checks actually executed. Verification of files, verification of finite examples, and proof of a theorem are different tasks. This guide keeps them separate.

1. Start with the portable publication checker

Extract the complete archive. Open a terminal in its top-level folder and run:

```
python3 -B code/verify_publication.py
```

Python 3.10 or newer is sufficient; this checker uses only the standard library. The publication run used Python 3.12.14. Do not use `-O` or `-OO`, which can disable assertions in historical code. The portable checker rejects an optimized interpreter.

The default command checks the exact moment normalization, interval moments, all five rational Chebyshev trial values plotted in the article, positive rational sufficient-degree parameters, sampling-grid arithmetic, Fejér coefficients and norms, the two explicit Riesz examples, and the publication dates. It does not claim to solve a continuous minimum by drawing a dense curve.

If `MANIFEST_SHA256.json` or `MANIFEST.json` exists in the package root, the checker verifies the SHA-256 digest of every listed file. Without a manifest it reports `not_present`; that status is not a successful integrity check. A manifest detects changes to the listed bytes. It does not establish the truth or novelty of a mathematical claim.

The script prints JSON and never overwrites publication files. To keep a new receipt without changing the preserved release, write it **outside** the extracted package:

```
python3 -B code/verify_publication.py > ../publication-check.json
```

Overwriting a manifest-listed receipt before verifying its hash would correctly produce a mismatch.

2. Rerun the selected round 6 suites

Run:

```
python3 -B code/verify_publication.py --run-selected > ../publication-selected-check.json
```

This makes a disposable copy of `evidence/history/round6/`, runs six original checkers there, compares their regenerated result objects to the archived JSON, and removes the temporary copy. The source evidence remains unchanged. The six scripts are:

Script: `branches_78_83_checks.py` **Principal scope:** Integer moments, operator identities, finite-box coefficients, and Chebyshev degree certificates

Script: `audit_83_quantum_checks.py` **Principal scope:** Separate rational degree certificates and exact Chebyshev trial values

Script: `branches_71_90_checks.py` **Principal scope:** Normalization, grid formulas, and twisted-ring examples

Script: `audit_90_spectral_checks.py` **Principal scope:** Independent integer matrix and closed-walk comparisons

Script: `quantum_72_77_checks.py` **Principal scope:** Exact finite checks behind the quantum-state comparisons

Script: `wavephysics_84_89_checks.py` **Principal scope:** Wave-model identities, Fermi–Dirac constants, weighted amplitudes, and the finite Riesz search

The last script enumerates all 262,143 nonempty subsets of $\{1, \dots, 18\}$ and compares 4,095 cases by a second autocorrelation method. The result 7/20 is the smallest normalized **Riesz trial quotient on that finite universe**, not a global minimum for unrestricted frequencies or a statement about the true K.

All six suites were rerun during publication preparation. They exited successfully, and all six regenerated JSON objects exactly matched the preserved results. The execution took approximately 16 seconds on the preparation machine; this is an execution record, not a promised runtime on another system. The default timeout is 180 seconds per script and can be adjusted, for example with `--timeout 600`.

See `evidence/current_verification.json` for the actual commands, interpreter, timestamps, outputs, result hashes, and status. Its initial run predates the final root manifest and explicitly records that the manifest was not yet present. Final package integrity can be checked with the default command after extraction.

3. Earlier suites and the historical evidence

Approaches 51–70 are under `evidence/history/round5/`. To repeat them manually, first copy that complete directory to a writable working folder, then run these commands in the **copy**:

```
python3 -B functions_51_56_checks.py
python3 -B branches_57_62_checks.py
python3 -B functions_63_68_checks.py
python3 -B root_69_70_checks.py
python3 -B audit_62_independent_checks.py
python3 -B audit_geometry_62_checks.py
```

These are the original available suite commands. This publication does not claim that this entire list was freshly rerun. The exact interval identities and selected hierarchy calculations were checked separately during the editorial review; that narrower scope is described in the mathematical audit.

The earlier unnumbered finite-search groundwork has its own independently recorded publication rerun. It covers all 4,194,303 nonempty subsets of $\{1, \dots, 22\}$, with two comparison implementations and continuous certificates for the winning example. Consult `evidence/groundwork_reproduction.md` and its receipts for the exact scope and strict bounds. To repeat that work, copy `evidence/history/previous_sources/`, provide NumPy (the recorded version is 2.3.5), and run `python3 run_verification.py` inside the copy. SciPy is not required by that runner. This guide does not repeat or enlarge that result.

Other historical exploratory programs can have different dependencies or assumptions. Their original notes remain with them; they are not silently modernized into a single uniformly tested software package. Approaches 41–50 were reconstructed from surviving descriptions. The missing approach-42 collection of 1,980 certificates has not been recreated. Neither a passing current checker nor an intact archive restores missing evidence.

4. Reading the evidence correctly

The mathematical appendix gives the proofs of unrestricted statements. A finite script checks the arithmetic of fixtures and identities; it does not prove every asserted theorem by enumeration. A valid sampled value is a

lower bound on negative depth, while exclusion of a deeper continuous minimum requires an upper certificate. Exact rational trial quotients are not automatically the optimized hierarchy values.

The article attributes the moment hierarchy to Lasserre and the polynomial-concentration architecture to prior work including Slot and Laurent. The novelty of the particular frequency-independent degree estimate is unverified. These internal mathematical and computational checks are not external peer review or formal proof-assistant verification. No claim is made that Chowla's conjecture has been resolved or that the research establishes a Bass Plus audio-performance improvement.

5. Rebuild the figures and reading editions

The canonical article sources are the Markdown files under `web/` and the mathematical appendix under `appendices/`. The publication preparation used Python 3.12.14, python-docx 1.2.0, NumPy 2.3.5, SciPy 1.17.0, Matplotlib 3.10.8, Pandoc 3.1.3, and LibreOfficeDev 26.8.0.0.alpha0. Exact historical checks have their own dependencies as stated above.

With the figure dependencies available, run `python3 code/generate_figures.py` to regenerate the supplied SVG, PNG, and JSON assets. With python-docx and Pandoc available, run `python3 code/build_documents.py` to rebuild both Word editions. These programs write generated publication outputs and create intermediate working files; use a working copy if you want the released manifest to remain unchanged.

The Word files contain native Office Math objects. The builder normalizes delimiter-property order, keeps displayed equations with their labels, and prints table entries as labeled records to avoid an observed table-import defect in the PDF converter. The canonical web sources retain their comparison tables. To export PDFs locally, run LibreOffice in headless conversion mode, for example:

```
soffice --headless --convert-to pdf --outdir documents
documents/BassPlus_Mathematics_of_Sound_Article.docx
soffice --headless --convert-to pdf --outdir documents
documents/BassPlus_Mathematics_of_Sound_Research_Edition.docx
```

Rendering is a separate check from mathematical verification. Inspect the exported pages for complete formulas, correct interval endpoints, visible ceilings and norms, readable tables, intact figure captions, and stable pagination. Different converter versions or fonts may change layout. The released PDFs were inspected visually; the corresponding handoff receipts identify the reviewed files by SHA-256.

Provenance, Evidence Boundaries, and Corrections

Prepared on 18 October 2026 for the *Mathematics of Sound* publication collection.

1. What “complete collection” means here

This package consolidates the research material available in the surviving workspace and the restored research journal. It includes the English publication text, a ninety-entry English research catalogue, mathematical appendices, code, exact and approximate computational records, audits, and the historical files from which the new text was prepared. It is complete with respect to those recovered materials. It is **not** a claim that every previously created temporary file has survived, or that missing data have been silently regenerated.

The original historical sources remain unedited in evidence/history/. This is important because an earlier candidate may be incorrect, a later audit may narrow its domain, and a later branch may reinterpret rather than replace an earlier result. The English catalogue reports the corrected scope. The historical record allows a reader to inspect how that scope was established.

This is an AI-assisted technical research report. “Proved in the notes” means an inspectable mathematical argument is supplied; “internally checked” means a separate check within this research process. Neither expression denotes external peer review, certification by a formal proof assistant, or established priority over the literature. No proof or disproof of the general Chowla cosine conjecture is claimed.

2. Surviving source families

The following counts describe the historical source directories before addition of publication-specific files. Manifest files and historical audit files are included in the counts.

Archive location: history/previous_sources/ **Scope:** Unnumbered groundwork and approaches 01–40 **Files:** 100 **Provenance:** Extracted from the preserved journal; every recorded byte count and SHA-256 hash matches

Archive location: history/recovered/ **Scope:** Approaches 41–50 **Files:** 7 **Provenance:** Four reconstructed mathematical texts, recovery notice, recreated checker, and its output

Archive location: history/round5/ **Scope:** Approaches 51–70 **Files:** 40 **Provenance:** Surviving separate notes, code, result files, audits, index, and manifest

Archive location: history/round6/ **Scope:** Approaches 71–90 **Files:** 43 **Provenance:** Surviving separate notes, code, result files, audits, illustration assets, index, and manifest

Archive location: history/Chowla_Research_Journal.txt **Scope:** Earlier consolidated journal with embedded sources **Files:** 1 **Provenance:** Preserved original byte sequence

The journal is 1,030,916 bytes. Its SHA-256 is:

9d81250bfbe7dff8b90078e76e2eabfb41b6897812051aa3bdc4435bf1eb0721

For publication preparation, the catalogue pass recalculated the hashes and byte counts of all 100 source files against the embedded journal registry. All 100 matched. It also compared the 39 listed non-manifest files in the round-five manifest and the 42 listed non-manifest files in the round-six manifest; no mismatches were found. These are **integrity checks**, not fresh mathematical experiment runs. A hash proves that a version is identifiable, not that its mathematics is correct.

The archive-wide inventory and final manifest describe the finished package. The machine-readable catalogue source map connects each numbered English entry to its source paths, byte counts, and source hashes. The history inventory inventories the copied historical material more broadly.

3. What was lost in approaches 41–50

The texts in history/recovered/ explicitly state that an earlier environment failure removed the original working files. They were reconstructed from surviving research statements. A reconstructed argument is available for scrutiny, but it is not the original source file, original execution log, or original certificate data.

The most consequential gap concerns **approach 42**, the proposed single-frequency local trap at $A_0 = \{1, 4, 5, 6\}$. The prior research reported 1,980 exact rational witnesses for replacement frequencies below 500, together with several Bernstein positivity certificates. Their original witness list and certificate files were lost. The surviving reconstruction gives the polynomial formulas, approximate location information, the reported earlier bounds, and a full analytic proof for every replacement frequency at least 500. It does not itself restore the finite witnesses. Therefore:

The infinite replacement tail is available as a complete reconstructed proof.

The prior finite-check success is preserved as a historical report.

The full local-trap theorem is not presented in the English catalogue as freshly machine-reproduced evidence.

Any future regeneration must be dated and stored as a new verification record, rather than replacing the provenance notice.

Other reconstructed branches describe earlier finite experiments whose original outputs are absent. Their general mathematical arguments are printed again, where available. In **approach 50**, a new integer checker and result file were actually recreated and survive; they verify the explicit denominator-81 and denominator-128 certificates. Those surviving outputs are different in status from the lost unsuccessful trial vectors at other denominators. Failure of one trial vector would not prove that a different certificate at the same denominator is impossible in any event.

Sources: recovery notice, biology reconstruction, and rounding reconstruction.

4. Corrections that the English publication must preserve

4.1. Finite and unrestricted frequency universes differ

Approach 13's relaxed mixed game has an unrestricted-domain value that cannot be asserted unchanged for every finite cutoff. The later clarification gives the finite-domain calculation separately. The catalogue cites both records and uses the corrected scope. A finite optimum elsewhere in the project likewise remains a theorem about that specified universe, not every positive integer frequency.

Source: game-domain clarification.

4.2. A nonnegative contact need not have exactly quadratic order

At a zero of a nonnegative analytic real function, the vanishing order is at least two and may be higher. Approach 23's boundary argument uses the weaker correct condition. The earlier wording was corrected and the correction retained in its audit.

Source: analysis audit.

4.3. Rational-factor counting has a factor of q^2 on both sides

For

$$H = 2m + \sum_{a \in A} (z^a + z^{-a}), q^2 H = \sum_j |R_j|^2,$$

with Gaussian-integer coefficient polynomials R_j , the total energy is $2mq^2$, while the sum of positive Fourier coefficients on the left is Nq^2 . The corrected estimate is

$$N \leq 2m^2 q^2 - m.$$

Its scale is $\sqrt{N}/q$ for the induced bound on m , not $\sqrt{N}/q^2$. The English discussion uses this normalization. Historical wording about a previously too-weak estimate remains in the reconstruction for traceability.

Sources: original denominator lemma, rounding reconstruction, and corrected denominator function.

4.4. Changing the upward shift changes the certificate question

For the seven-frequency example $\{1, 2, 3, 5, 6, 7, 8\}$, denominator one is impossible at $m=2$, but an explicit certificate exists at $m=3$. The smaller-shift obstruction is not a counterexample to a statement that explicitly requires an additional positivity margin. The catalogue also distinguishes existence of one rational factor from existence of a finite sum of rational squared factors.

Sources: phase/dilation distinctions and shift distinction.

4.5. A failed detector is not a small true minimum

Several branches prove upper bounds on the effectiveness of a particular lower certificate: fixed-degree moments, residue averages, ordinary-majority conditioning, fixed quantiles, zero-threshold Fermi–Dirac weights, or a specified envelope. These do not prove corresponding upper bounds on $K(A)$. Consecutive frequencies are repeatedly useful precisely because they have a large true negative excursion that a weak detector misses.

Similarly, the finite minimum $7/20$ in approach 87 concerns its Riesz-product certificate normalized by $\sqrt{N}$ over subsets of $\{1, \dots, 18\}$. It is not the unrestricted minimum of that certificate and not the extremal Chowla constant. General positivity of that raw detector was not separately proved in the source.

Source: Riesz-product scope.

4.6. Approximation is not the conjectured growth theorem

Approaches 08, 56, 62, and 83 provide ways to preserve or approximate a minimum. Their bounds do not prove that the minimum is at least a fixed multiple of $\sqrt{N}$. The Chebyshev improvement concerns a sufficient polynomial degree in this project's approximation argument. Without a complete literature comparison it must not be advertised as a new optimal method or a faster best-known algorithm. Its proof uses the unknown edge to exhibit a candidate polynomial; the moment optimization defining the certificate does not require that edge as an input.

Sources: moment hierarchy, Chebyshev filter, and filter audit.

4.7. Sampling does not certify the unsampled minimum without an error argument

A rational witness proves a lower bound for depth at one point. An approximate grid minimum is not by itself a continuous minimum. The grid example in approach 90 retains exact mean square and sufficient interpolation information while missing a much deeper excursion between nodes. The static plot is illustrative; the analytic identity and exact finite checks establish its stated claim.

Sources: continuous grid certificate and phase-grid counterexample.

5. Relationship between branches

The ninety entries were produced in successive rounds of hypothesis generation and checking. Their numbering measures bookkeeping, not scientific independence. The following connections are particularly important when describing the work publicly.

Recurrent mechanism: Frequency relations and compactness **Representative branches:** 08, 30, 39, 51, 56
What changes between versions: Attainment, descent structure, random torus coding, extremal functions, and effective finite approximation

Recurrent mechanism: Polynomial spectral filters **Representative branches:** 19, 47, 62, 83 **What changes between versions:** Variable of the polynomial, approximation target, and sufficient degree

Recurrent mechanism: Positive rational factorization **Representative branches:** 26, 31–33, 40, 47–50, 67
What changes between versions: Arithmetic constraints, shifts, finite decision bounds, helper factors, rounding, and denominator estimates

Recurrent mechanism: Gibbs and information weighting **Representative branches:** 03, 17, 35, 46, 63, 66, 82, 87, 88 **What changes between versions:** Proposed zero information, inverse problems, choice of weight, entropy budget, and saturation

Recurrent mechanism: Spectral and graph language **Representative branches:** 01, 02, 14, 61, 78–83, 90
What changes between versions: Commuting versus free operators, finite compression, boundary conditions, and lower-edge filters

Recurrent mechanism: Phase information and averaging **Representative branches:** 09, 12, 41, 45, 68, 72–74, 86 **What changes between versions:** Whether the original common-time orbit and signed mean are preserved

Recurrent mechanism: Positive weighted profiles **Representative branches:** 11, 16, 25, 27–29, 44, 70, 89
What changes between versions: Attempts to transfer weighted positivity to distinct equal-amplitude frequencies

The same counterexample can legitimately illuminate two different physical descriptions, but should not be counted twice as a new mathematical discovery. The identical-intensity example in 72 and 86 is an explicit instance. Standard constructions such as Fejér kernels, Chebyshev polynomials, Riesz products, and tight-binding operators are attributed in the full articles rather than being renamed as original methods.

6. Mathematical proof, exact arithmetic, and numerical exploration

The historical code contains more than one evidence type:

General mathematical arguments establish claims for infinite families or every set satisfying stated hypotheses.

Exact certificates use integer identities, rational intervals, or polynomial positivity decompositions to certify specified finite statements.

Exhaustive finite searches establish claims only in the enumerated universe, with their correctness depending on the enumeration and certificate logic.

Floating-point searches and plots locate candidates or illustrate behavior; they do not themselves certify unrestricted conclusions.

An exact negative value is a valid lower bound for K , even if no global minimum was computed. Conversely, showing positivity of $k + f_A$ on the entire circle gives an upper bound for K . Both directions must be kept distinct. A proof using a rationally certified value found first by a numerical search is valid because the final certificate no longer depends on the numerical optimizer's correctness.

A separate publication verification report records the selected scripts actually rerun while assembling the English material. It should be consulted for current environment details and execution results. This provenance note deliberately does not claim that all original experiments were rerun. Re-running a program in a disposable copy is preferable to overwriting an archival result file whose earlier hash and date are part of the evidence.

7. Physical and product scope

The negative crest factor is an exact identity for the stated equal-amplitude signal. The lattice operator is an exact auxiliary mathematical model. Quantum oscillator examples show what mean motion and measurement variance do or do not constrain. None of these facts alone demonstrates a new law of acoustics, negative physical energy, a quantum behavior of the Bass Plus application, or improved audible output.

The publication may explain why sound motivates the investigation and why missed peaks or lost phase information matter conceptually. Any future product claim needs its own implemented method, applicable signal assumptions, and relevant measurements. Arbitrary equalizer gains, phases, transient recordings, and noninteger frequencies are not automatically covered by the equal-unit-amplitude, full-period conjecture studied here.

8. Recommended language for future citations

Suitable: "The report records ninety tested research branches, including explicit counterexamples to proposed shortcuts and frequency-independent approximation bounds under stated assumptions."

Suitable: "An AI-assisted internal check found no remaining error in the documented scope; no external peer review or formal verification is claimed."

Unsuitable: "Ninety new mathematical discoveries," "Chowla's conjecture was solved," "the best-known algorithm was improved," or "quantum physics now improves the app's sound." Those stronger statements have not been established by this collection.

The corrected catalogue and full appendices should be cited for current claims. Historical hypotheses should be quoted as historical hypotheses, with their later correction or rejection attached.

Publication Status and Claim Boundaries

Version 1.0, 18 October 2026. This is an illustrated mathematical research report and a reproducibility collection prepared for the Mathematics of Sound section of the Bass Plus website.

What the package establishes

The mathematical appendix supplies full proofs for the results promoted in the current report, including the normalization, phase-mass estimate, fixed-size attainment, moment hierarchy specialization, explicit Chebyshev sufficient degree, spectral identities, finite-box error, grid counterexample, grid discretization bounds, weighted-amplitude obstruction, and several limitations of proposed detector functionals. Exact certificates and finite enumeration records support the stated examples on their declared domains.

What remains open

The package neither proves nor disproves the existence of an absolute positive constant in Chowla's square-root lower bound. It does not identify the best universal constant. Values from finite searches, including $7/20$ for a specific Riesz-product quotient on subsets of $\{1, \dots, 18\}$, are not promoted to universal theorems.

Attribution and priority

The polynomial quotient hierarchy is a specialization of Lasserre's established pushforward measure hierarchy. Chebyshev and needle-polynomial concentration methods are established prior art, including work by Slot and Laurent. The appendix proves explicit constants for a specialization uniform in the largest frequency, but priority for that specialization has not been established. No result is marketed as a newly proved famous conjecture or as an independently verified OpenAI discovery.

Review status

The materials were checked through derivations, exact arithmetic, selected repeated computations, and review within the same ChatGPT-assisted workflow. They have not received external peer review and were not formally verified in Lean, Coq, or another proof assistant. Code checks certify their specified computations, not every mathematical sentence in the research history.

Historical completeness

All currently available research source files are included. The original bytes of approaches 41–50 were not available; explicit reconstructions are supplied. In particular, the earlier set of 1,980 certificates associated with approach 42 is missing. Historical notes are preserved for traceability and can contain superseded statements. The curated English report, catalogue, and corrections take precedence over those older statements.

Authorship and publication

Prepared as CliDeck Research Notes, initiated by the Bass Plus project owner with ChatGPT assistance for research, writing, calculations, and code. No unprovided personal name, institutional affiliation, or third-party endorsement is assigned. This package is a publication handoff: preparing it has not changed the live Bass Plus website. The accompanying Codex instructions specify the website publication task.